Suzanne Crifo | Research Statement

My research interests and experiences lie in algebra and combinatorics. More specifically, my dissertation is on affine Lie algebras and their representations. I am interested in learning more about the structure of certain representations by determining key features of a representation called maximal dominant weights and their multiplicities. Below, I will give some background into affine Lie algebras and their representations. I will then discuss my contributions to this area as well as give possible future directions and projects with undergraduates.

Lie Algebras and their Representations

Sophus Lie discovered Lie algebras in the 19th century while studying the symmetries of solutions of differential equations. Today Lie algebras influence the study of differential geometry and topics in mathematical physics including quantum mechanics and particle physics.

A Lie algebra is a vector space, $\mathfrak{g}$, together with a bilinear product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies $[x,x] = 0$ for all $x \in \mathfrak{g}$ and the Leibniz identity, $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ [4]. Note that this product is skew symmetric, i.e. $[x,y] = -[y,x]$, due to the first condition and the bilinearity of the product. For example, if V is a vector space, the set of linear operators on V is a Lie algebra, denoted $\mathfrak{gl}(V)$ with the bracket $[x,y] = x \circ y - y \circ x$ where $x \circ y$ is the composition of linear operators. If V has finite dimension n, the elements of this Lie algebra can be thought of as $n \times n$ matrices. In the case $n = 2$, the subalgebra of $\mathfrak{gl}(2)$ consisting of matrices of trace 0 is an important Lie algebra denoted $\mathfrak{sl}(2)$. The basis vectors and corresponding bracket relations for this space are

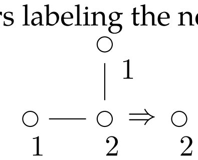
$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } [h,e] = 2e, [h,f] = -2f, [e,f] = h$$

An ideal of a Lie algebra is a subspace I of L such that $[x,y] \in I$ for all $x \in L$ and $y \in I$. A Lie algebra is called simple if it is nonabelian (meaning $[x,y] \neq 0$ for some elements $x, y \in L$) and has no nontrivial proper ideals. $\mathfrak{sl}(2,\mathbb{F})$ is a simple Lie algebra, provided $\mathrm{char}\mathbb{F} \neq 2$. A Lie algebra is called semisimple if it is the direct sum of simple Lie algebras.

A representation is a Lie algebra homomorphism $\rho : \mathfrak{g} \to gl(V)$, where V is a vector space and $\rho([x,y])(v) = (\rho(x)\rho(y) - \rho(y)\rho(x))v$ for all $x, y \in \mathfrak{g}$ and $v \in V$. V is then referred to as a $\mathfrak{g}$-module; the words "representation" and "module" are often used interchangeably. Finite dimensional simple Lie algebras and their representations have been studied extensively. An affine Lie algebra is an infinite-dimensional analog of a finite dimensional, semisimple Lie algebra. Therefore, it is natural to study the representation theory of affine Lie algebras.

An affine Lie algebra can be defined in terms of generators and relations determined by a generalized Cartan matrix (GCM) and corresponding Dynkin diagram. For example, the GCM and Dynkin diagram corresponding to the affine Lie algebra $B_3^{(1)}$ are shown below. Notice that if you were to create a vector, $\mathbf{a}$, from the numbers labeling the nodes of the diagram, that $A \cdot \mathbf{a} = 0$.

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$



A "realization" of a GCM is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where the Cartan subalgebra, $\mathfrak{h}$, is a complex vector space of dimension $n+2$, $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ the set of simple roots, and $\Pi^\vee = \{h_0, h_1, \ldots, h_n\} \subset \mathfrak{h}$ the set of simple coroots such that Π and $\Pi^\vee$ are linearly independent and $\alpha_j(h_i) = a_{ij}$ where the a_{ij} are entries of the GCM [6]. A root system can be defined independently of Lie algebras, and is a finite subset of an euclidean space satisfying certain axioms [4]. The affine Lie algebra associated with a realization has generators e_i, f_i ($i \in \{0, 1, \ldots, n\} = I$) and $\mathfrak{h}$ and is defined by the relations:

- $[e_i, f_j] = \delta_{ij} h_i$ for $i, j \in I$
- $[h, h'] = 0$ for all $h, h' \in \mathfrak{h}$
- $[h_i, e_j] = a_{ij} e_j$ for $i, j \in I$
- $[h_i, f_j] = -a_{ij} f_j$ for $i, j \in I$
- $(ad(e_i))^{1-a_{ij}} e_j = 0$ for $i \neq j$
- $(ad(f_i))^{1-a_{ij}} f_j = 0$ for $i \neq j$

Weight Modules and Maximal Dominant Weights

A $\mathfrak{g}$-module V is called a "weight module" if it admits a weight space decomposition $V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu$ where $V_\mu = \{v \in V | h \cdot v = \mu(h) v \; \forall \; h \in \mathfrak{h}\}$. $\mu \in \mathfrak{h}^*$ is called a "weight" if $V_\mu \neq \{0\}$. That is, the representation of a Lie algebra is a way to view each element of the Lie algebra as a linear transformation of a suitable vector space. The weights of the Lie algebra are the eigenvalues of the linear transformations acting on the vector space. The dimension of the corresponding eigenspaces are called the weight multiplicities.

My research involves a specific type of representation, $V(\Lambda)$, called an integrable highest weight module. A weight in $V(\Lambda)$ is called maximal if $\lambda + \delta \notin P(\Lambda)$, where the null root, $\delta = \sum_{i=0}^{n} a_i \alpha_i$ for a_i the coordinates of the null vector $\mathbf{a}$. A weight, λ, is dominant integral if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$. To describe the structure of $V(\Lambda)$, one only needs to describe its maximal dominant weights. The maximal dominant weights form a type of "ceiling" to the set of all weights, since you can obtain all other weights by subtracting non-negative integer multiples of δ. Any other weight lies on such an infinite string off of a maximal dominant weight.

It is known there are finitely many maximal dominant weights for any integrable highest weight representation of an affine Lie algebra. However, determining these maximal dominant weights is a nontrivial task. Tsuchioka found all maximal dominant weights for modules of the form $V(\Lambda_0 + \Lambda_s)$ where $0 \leq s < p$ for type $A_{p-1}^{(1)}$ [8]. This work was generalized in 2014 by Jayne and Misra, who found all maximal dominant weights for modules of the form $V((k-1)\Lambda_0 + \Lambda_s)$ where $0 \leq s \leq n-1$ for type $A_{n-1}^{(1)}$ [5]. In 2017, Kim, Lee, and Oh determined all maximal dominant weights for $V(\Lambda)$ where Λ is of level 2 for types $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$, and of level 1 for $C_n^{(1)}$ [7].

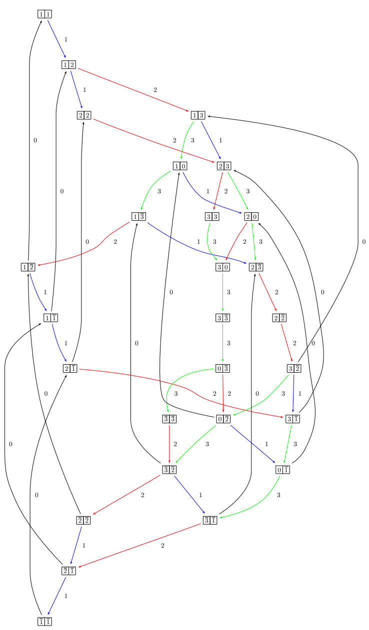
In my thesis, I have given explicit descriptions of maximal dominant weights for the integrable highest weight representation of any affine Lie algebra with highest weight $k\Lambda_0$. This corresponds with Kim, Lee, and Oh's paper in the case of $2\Lambda_0$. The following describes the maximal dominant weights for $V(k\Lambda_0)$ for $\mathfrak{g} = B_n^{(1)}$.

Theorem. *[2] Let $n \geq 3$, $\Lambda = k\Lambda_0, k \geq 2$. Then*
$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - \Big(\sum_{i=3}^{n} (2l - (x_2 + \sum_{j=3}^{i} l_j))\alpha_i \Big)$
where

- $1 \leq x_2 \leq k$
- $l = \max\{x_1, \lceil \frac{x_n}{2} \rceil\}$
- $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor$
- $0 \leq l_3 \leq \lfloor \frac{x_2}{2} \rfloor - l_1$
- $0 \leq l_n \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$
- $l_2 = x_2 - x_1$ *for* $n = 3, l_2 = 0$ *else*

Weight Multiplicities

The identification of the multiplicities of the maximal dominant weights is an important part of determining the structure of a representation. In my thesis, I use a combinatorial tool called a "crystal base," which was developed in 1990 by Kashiwara and Lusztig. In particular, I use the relations in perfect crystals, like the perfect crystal B_2 shown below for the affine Lie algebra $B_3^{(1)}$, to explore the path realization of a representation. This path realization allows one to count multiplicities of weights, using a formula called an energy function [3]. I am currently working on determining multiplicities of specific maximal dominant weights by fixing my values for n and k. For example, the following table lists all maximal dominant weights for the $B_n^{(1)}$-module $V(2\Lambda_0)$ where $n = 3, 4$, and 5. I hope to generalize my results in the future.



n	Maximal Dominant Weight	Multiplicity
3	$2\Lambda_0$	1
3	$\Lambda_2 - \delta$	1
3	$\Lambda_0 + \Lambda_1 - \delta$	1
3	$2\Lambda_3 - 2\delta$	3
3	$2\Lambda_1 - 2\delta$	3
4	$2\Lambda_0$	1
4	$\Lambda_2 - \delta$	1
4	$\Lambda_0 + \Lambda_1 - \delta$	1
4	$\Lambda_3 - 2\delta$	3
4	$2\Lambda_4 - 2\delta$	3
4	$2\Lambda_1 - 2\delta$	4
5	$2\Lambda_0$	1
5	$\Lambda_2 - \delta$	1
5	$\Lambda_0 + \Lambda_1 - \delta$	1
5	$\Lambda_3 - 2\delta$	3
5	$\Lambda_4 - 2\delta$	3
5	$2\Lambda_5 - 3\delta$	10
5	$2\Lambda_1 - 2\delta$	5

Directions for Undergraduate Research

Because of the computational and combinatorial nature of my current work, I plan to involve advanced undergraduate students in this process. Students will work with SageMath, an open-source mathematics software system, to investigate specific crystals and find patterns. The problem of finding more efficient ways to study these crystals could interest students.

Additionally, because of my coursework in Operations Research, I am eager to pursue optimization problems that interest undergraduate students. My experience with both linear and dynamic programming, and soon nonlinear programming, gives me a variety of tools that I can introduce to students. These techniques can be applied to specific problems that interest individual students. Undergraduates will be excited by the opportunity to solve real world problems.

References

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