Solutions to Optional Written Homework 6

① $\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 2 \\ 0 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = (2-\lambda)\big((2-\lambda)(-\lambda)+1\big)$

$= (2-\lambda)(\lambda^2-2\lambda+1)$

$= (2-\lambda)(\lambda-1)^2$

Eigenvalues: 2, 1 ← alg. mult 2.

E_2: $(A-2I)v=0 \Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore E_2 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$x_2 = 0$

$x_3 = 0$

E_1: $(A-I)v=0 \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + 3x_3 = 0$

$x_2 - x_3 = 0$

$E_1 = \operatorname{span}\left\{\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}\right\}$

geometric multiplicity 1.

A is not diagonalizable since for eigenvalue 1, alg. mult $\neq$ geom. mult.

② a) $T\big(\lambda(a_1x^2+b_1x+c_1) + (a_2x^2+b_2x+c_2)\big) = T\big((\lambda a_1+a_2)x^2 + (\lambda b_1+b_2)x + (\lambda c_1+c_2)\big)$

$= \big((\lambda a_1+a_2)-(\lambda b_1+b_2)-(\lambda c_1+c_2)\big)x^2$

$+\big((\lambda b_1+b_2) - 4(\lambda c_1+c_2)\big)x$

$+ -2(\lambda b_1+b_2) - (\lambda c_1+c_2)$

$= \lambda(a_1-b_1-c_1)x^2 + (a_2-b_2-c_2)x^2$

$+ \lambda(b_1-4c_1)x + (b_2-4c_2)x$

$+ \lambda(-2b_1-c_1) + -2b_2-c_2$

$= \lambda\big[(a_1-b_1-c_1)x^2 + (b_1-4c_1)x + (-2b_1-c_1)\big]$

$+(a_2-b_2-c_2)x^2 + (b_2-4c_2)x + -2b_2-c_2$

$= \lambda T(a_1x^2+b_1x+c_1) + T(a_2x^2+b_2x+c_2)$.

So T is a linear transformation.

b) $\mathcal{B} = \{x^2, x, 1\}$

$M(T) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -4 \\ 0 & -2 & -1 \end{bmatrix}$

c) $\to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\ker(T) = \{0\} = 0x^2+0x+0$.

$\operatorname{Range}(T) = P_2$.

d). T is both injective and surjective.

e) $\det \begin{bmatrix} 1-\lambda & -1 & -1 \\ 0 & 1-\lambda & -4 \\ 0 & -2 & -1-\lambda \end{bmatrix} = (1-\lambda)\big((1-\lambda)(-1-\lambda)-8\big)$

$= (1-\lambda)(\lambda^2-9) = (1-\lambda)(\lambda-3)(\lambda+3)$

E_1: $\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -4 \\ 0 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $E_1 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$\boxed{\text{Eigenvalue: } 1 \quad \text{Eigenvector: } x^2}$

E_3:
$$\begin{bmatrix} -2 & -1 & -1 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}\right\}$$

$\boxed{\begin{array}{l}\text{Eigenvalue: } 3 \\ \text{Eigenvector: } x^2-4x+2\end{array}}$

E_{-3}:
$$\begin{bmatrix} 4 & -1 & -1 \\ 0 & 4 & -4 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{-3} = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right\}$$

$\boxed{\begin{array}{l}\text{Eigenvalue: } -3 \\ \text{Eigenvector: } x^2+2x+2\end{array}}$

f) yes, 3 distinct Eigenvalues.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

3

1. $\langle A, A \rangle = \text{trace}(A^T A) = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$ and $= 0 \iff a_{11} = a_{12} = a_{21} = a_{22} = 0$.

2. $$\begin{aligned} \langle \lambda A + B, C \rangle &= \text{trace}(C^T(\lambda A + B)) \\ &= \text{trace}(C(\lambda A + B)^T) \\ &= \text{trace}(\lambda C A^T + C B^T) \\ &= \lambda\, \text{trace}(C A^T) + \text{trace}(C B^T) \\ &= \lambda\, \text{trace}(C^T A) + \text{trace}(C^T B) \\ &= \lambda \langle A, C \rangle + \langle B, C \rangle \end{aligned}$$

Properties of trace + transpose <u>OR</u> can do explicitly with entries of matrices.

3. $\langle A, B \rangle = \text{trace}(B^T A) = \text{trace}((B^T A)^T) = \text{trace}(A^T B) = \langle B, A \rangle$.

4

$$1 + x - 2x^2 = \frac{\langle 1, 1 + x - 2x^2 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle x, 1 + x - 2x^2 \rangle}{\langle x, x \rangle} x + \frac{\langle \frac{1}{2}(3x^2-1), 1 + x - 2x^2 \rangle}{\langle \frac{1}{2}(3x^2-1), (\frac{1}{2}(3x^2-1)) \rangle} \left(\frac{1}{2}(3x^2-1) \right)$$

$$= \frac{1}{2} \int_{-1}^{1} 1 + x - 2x^2 \, dx \, (1) + \frac{3}{2} \int_{-1}^{1} x + x^2 - 2x^3 \, dx \, (x) + \frac{5}{2} \int_{-1}^{1} \frac{1}{2}(3x^2-1)(1+x-2x^2) \, dx \left(\frac{1}{2}(3x^2-1) \right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{2}{3} - \left(-1 + \frac{1}{2} + \frac{2}{3}\right)\right)(1) + \frac{3}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{2}\right)\right)x + \frac{5}{4}\left(-\frac{6}{5} + \frac{5}{3} + \frac{3}{4} - \frac{1}{2} - 1 - \left(\frac{6}{5} - \frac{5}{3} + \frac{3}{4} - \frac{1}{2} + 1\right)\right)\left(\frac{1}{2}(3x^2-1)\right)$$

$$= \frac{1}{3}(1) + (1)x + -\frac{4}{3}\left(\frac{1}{2}(3x^2-1)\right)$$