MA 405-002: Introduction to Linear Algebra and Matrices, NCSU, Spring 2018

Study Guide for Test #3

Operations with Linear Maps

What are the different ways you can combine linear maps?

How does this connect to other concepts we've learned in this class?

Kernel and Image

How do the kernel and image of linear transformation connect to other concepts we've learned in this class?

What can you say about the dimensions of the kernel and image of a linear map?

List everything you know about the image and kernel of a linear map below. How do you find bases for each? What are their dimensions? Where do they live? What other questions do they answer?

Image	**Kernel**

Injectivity and Surjectivity

What does it mean for any map to be injective? What about surjective?

How can you determine if a map is injective? What about surjective?

Isomorphisms

What is an isomorphism?

How do isomorphisms relate to other topics we've covered in this class?

Name all the vector spaces you can find that are isomorphic to $\mathbb{R}^6$. Give the isomorphism.

Invertibility

What does it mean for a map to be invertible? What about a matrix?

Try some of the operations you mentioned on maps. Do they preserve invertibility?

Eigenvalues, Eigenvectors, and Determinants

What is an eigenvalue? An eigenvector?

What is the determinant of a matrix?

How do all of these relate?

What are the properties of determinants?

How do determinants relate to other topics we've studied in this class?

How do eigenvalues and eigenvectors relate to other topics?

Practice Problems

1. Let $L : V \to W$ be a linear transformation between finite dimensional vector spaces V and W and let A be the matrix of L with respect to a choice of bases for V and W.
   (a) What does the number of free variables tell us about L?

   (b) What does the number of basic variables tell us about L?

2. Suppose the matrix of $L : V \to W$ is
$$\begin{bmatrix} 0 & 1 & -2 & 3 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$
   Determine $\dim(\ker(L))$ and $\dim(\mathrm{Im}(L))$.
3. Let $L : V \to W$ be a linear transformations between vector spaces V and W. Define the kernel of L, $\ker(L)$.
4. Prove that $\ker(L)$ is subspace of V.

5. Find a parametrization of the set of solutions to the matrix equation

$$\begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 0 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

6. Let $S : U \to V$, $T : V \to W$ be linear transformations of finite dimensional vector spaces. If A and B are the matrices of S and T respectively, what is the matrix of $T \circ S$?

7. Suppose that V is a vector space of dimension n and let α be a basis for V. Let $I : V \to V$ denote the identity transformation of V. What is the matrix $[I]_\alpha^\alpha$ of the identity transformation with respect to α?

8. Let $L : V \to W$ be a linear transformation between finite dimensional vector spaces V and W. Define what it means for L to be invertible.

9. Prove that L is invertible if it is one-to-one and onto.

10. Show that if B is obtained from A by multiplying the first row of A by a scalar, c, and adding it to the second row (i.e. $A \xrightarrow{cR_1+R_2} B$), then $\det(A) = \det(B)$.

11. Prove that if one row of a matrix A is a multiple of another row of A, then $\det(A) = 0$.

12. What is ijth cofactor of an $n \times n$ matrix A and what is the formula for $\det(A)$ in terms of the cofactors for the ith row of A?

13. Use the formula for the second row of A to compute $\det(A)$ by this formula for the following matrix A.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{bmatrix}$$

14. Compute A^{-1} for

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{bmatrix}$$

15. Suppose the matrix B is obtained from the square matrix A by interchanging two rows of A. What do we know about $\det(A)$ and $\det(B)$?

16. Suppose that A and B are 2×2 matrices. Is it always the case that $AB = BA$? If so, why? If not, provide a counterexample.

17. Suppose that A and B are 2×2 matrices. Is it always the case that $\det(AB) = \det(BA)$? If so, why? If not, provide a counterexample.

18. (25 pts) Let $S : U \to V$ and $T : V \to W$ be linear transformations of finite dimensional vectors spaces.

    (a) Define what it means for S to be injective or one-to-one.

(b) How can you tell that a linear transformation is injective from its matrix? (*Hint:* This isn't about determinants since the matrix might not be square.)

(c) Let $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrices with respect to the standard bases:

$$[S]_{st}^{st} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad [T]_{st}^{st} = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & -2 \end{bmatrix}$$

Use these matrices and their product to determine whether S, T and TS are injective.

19. Let A be an $n \times n$ matrix.

(a) What is the ijth minor of A?

(b) What is the formula for the $\det(A)$ in terms of the minors for the ith row of A?

(c) Suppose that A is an upper triangular matrix. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

Use the formula from (b) and mathematical induction to prove that $\det(A) = a_{11}a_{22}\dots a_{nn}$. (*Hint:* How should the formula be applied?)

20. Suppose that $L : \mathbb{R}^7 \to \mathbb{R}^4$. What do we know about the kernel of L?

21. Suppose A is an $n \times n$ matrix and $\det(A) = 0$. What do we know about the kernel of A?

22. Suppose that the matrix of $L : \mathbb{R}^4 \to \mathbb{R}^4$ with respect to the standard basis is

$$[L]_{st}^{st} = \begin{bmatrix} 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

What is the dimension of the image of L?

23. Let $S : U \to V$ and $T : V \to W$ be linear transformations of finite dimensional vectors spaces.

(a) Prove that $TS : U \to W$ is a linear transformation.

(b) Prove that $\ker(S) \subset \ker(TS)$.

(c) What does (b) say about the number of free variables for the matrices of S and for TS?

24. Let A be an $n \times n$ matrix.

(a) What is the ijth cofactor of A?

(b) What is the formula for the $\det(A)$ in terms of the cofactors for the ith row of A?

(c) Use the formula from (b) to calculate the $\det(A)$ twice using two different rows of A.

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}$$