Solutions to Written Homework 5

① a) Not an isomorphism since T is not injective $\to$ $\ker(T) = \operatorname{span}\{1\}$

b) yes $\to$ 3 pivots $\Rightarrow$ isomorphism

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Then $T^{-1}: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y-z, -y+z, z)$

② a) $V = \mathbb{R}^4 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$ $\quad W = \{ p \in P_4(\mathbb{R}) \mid p(0) = 0 \} = \{ ax^4 + bx^3 + cx^2 + dx \}$

$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = ax^4 + bx^3 + cx^2 + dx$

b) $V = P_5(\mathbb{R}) = \{ ax^5 + bx^4 + cx^3 + dx^2 + ex + f \mid a,b,c,d,e,f \in \mathbb{R} \}$ $\quad W = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \,\middle|\, a,b,c,d,e,f \in \mathbb{R} \right\}$

$T(ax^5 + bx^4 + cx^3 + dx^2 + ex + f) = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$

c) Not isomorphic. $\dim(\mathbb{R}^3) = 3$, $\dim(P(\mathbb{R})) = \infty$. $\nexists\, T$ surjective.

③ a) $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right)$ $\quad A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

b) $\left(\begin{array}{cc|cc} \cos(\theta) & -\sin(\theta) & 1 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -\tan(\theta) & \frac{1}{\cos\theta} & 0 \\ 0 & \frac{1}{\cos\theta} & -\tan\theta & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right)$

c) $\left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$ $\quad A = A^{-1}$!

④ $\Leftarrow$ Assume both S and T are invertible. Then $(ST)T^{-1}S^{-1} = S(TT^{-1})S^{-1} = SIS^{-1} = I$. So ST is invertible.

$\Rightarrow$ Assume ST is invertible. We want to show both S and T are invertible.

ST invertible $\Rightarrow$ ST injective ($\ker(ST) = \{0\}$) and ST surjective ($\operatorname{Im}(ST) = V$).

$ST = S \circ T \Rightarrow ST(v) = S(T(v)) \Rightarrow$ if $T(v) = 0$, $S(T(v)) = 0$ since $S(0) = 0$.

Then $\ker(T) \subset \ker(ST) = \{0\}$. Since V is finite dimensional, $\operatorname{Im}(T) = V$.

Then T is an isomorphism.

Similarly, since $ST(v) = S(T(v))$ then $\forall\, w = ST(v)$, $w = S(T(v)) \Rightarrow \operatorname{Im}(ST) \subset \operatorname{Im}(S)$.

Then $V \subset \operatorname{Im}(S) \subset V \Rightarrow \operatorname{Im}(S) = V$ so $\ker(S) = \{0\}$ since V finite dimensional

$\Rightarrow S$ is an isomorphism (Verify anything that is not clear to you! There are most likely other ways to prove this.)

⑤ a) Let $T = I: V \to V$ the identity isomorphism: $I(v) = v\ \forall v \in V$. Then $V \cong V$.

b) Assume $V \cong W$. Then $\exists\, T: V \to W$ that is linear, injective and surjective. Then T is invertible. That is, $\exists\, T^{-1}: W \to V$ s.t. $TT^{-1} = \mathrm{Id}_W$ and $T^{-1}T = \mathrm{Id}_V$. Then T^{-1} is an isomorphism from W to V so $W \cong V$.

c). Assume $U \cong V$ and $V \cong W$.

Then $\exists$ $S: U \to V$ and $T: V \to W$ both linear, injective, and surjective.

Then both S and T are invertible $\Rightarrow \exists$ $S^{-1}: V \to U$ and $T^{-1}: W \to V$

s.t. $SS^{-1} = Id_V$, $S^{-1}S = Id_U$, $TT^{-1} = Id_W$ $T^{-1}T = Id_V$

Then $TS: U \to W$ is an isomorphism with inverse $S^{-1}T^{-1}$. So $U \cong W$

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a) $\begin{pmatrix} -1 & 3 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ -20 \end{pmatrix} = -5x \Rightarrow x$ is an eigenvector with eigenvalue -5.

b) $T(x^3) = x(3x^2) - 4(x^3) = -x^3 \Rightarrow x^3$ is an eigenvector with eigenvalue -1.

c) $T(e^{4x}) = 16e^{4x} + e^{4x} = 17e^{4x} \Rightarrow e^{4x}$ is an eigenvector with eigenvalue 17.