MA 405-002: Introduction to Linear Algebra and Matrices, NCSU, Spring 2018

Written Homework #5

Due: Wednesday, April 11

Please indicate if you would like the grader to give feedback regarding your mathematical writing. This feedback will not be factored into your grade and only serves to help you grow as a mathematician.

1. Are the following transformations isomorphisms? If so, find the inverse transformation:
   (a) $T : P_3 \to P_3$ given by $T(p(x)) = x\frac{dp}{dx}$.
   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect to the standard basis is
   $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Are the following pairs of vector spaces isomorphic? If so, find an isomorphism $T : V \to W$. If not, say why not.
   (a) $V = \mathbb{R}^4, W = \{p \in P_4(\mathbb{R}) | p(0) = 0\}$
   (b) $V = P_5(\mathbb{R}), W = M_{2\times 3}(\mathbb{R})$
   (c) $V = \mathbb{R}^3, W = P(\mathbb{R}) = \{\text{all polynomials over } \mathbb{R}\}$

3. Find the inverses of the following matrices.
   (a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. Axler Section 3D number 9: Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

5. Show that isomorphism of vector spaces is an equivalence relation. That is, prove each of the following statements:
   (a) Every vector space V is isomorphic to itself.
   (b) If V and W are vector spaces, and V is isomorphic to W, then W is isomorphic to V.
   (c) If U, V, and W are vector spaces and U is isomorphic to V and V is isomorphic to W, then U is isomorphic to W.
   (*Hint:* In each case the key point is to find the isomorphism in terms of the other information given.)

6. Verify that the given vector is an eigenvector of the given mapping, and find the eigenvalue.
   (a) $x = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \in \mathbb{R}^2, T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A = \begin{pmatrix} -1 & 3 \\ 0 & -5 \end{pmatrix}$
   (b) $p = x^3 \in P_3(\mathbb{R}), T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p) = xp' - 4p$
   (c) $f = e^{4x} \in V = \{\text{functions which have derivatives of all orders}\}, T : V \to V$ defined by $T(f) = f'' + f$