Solutions to Written Homework #4

① Let $T: \mathbb{R}^n \to \mathbb{R}$ be a linear map. Recall that $\mathrm{Im}(T)$ is a subspace of $\mathbb{R}$. Since $\dim(\mathbb{R})=1$, any subspace of $\mathbb{R}$ has dim 0 or 1.
If $\dim(\mathrm{Im}(T))=1$ then $\mathrm{Im}(T)=\mathbb{R}$ and T is surjective
If $\dim(\mathrm{Im}(T))=0$ then $\mathrm{Im}(T)=\{0\}$ so T is the zero map. This is the only map from $\mathbb{R}^n$ to $\mathbb{R}$ that is not surjective.

② a) shortcut to prove both additivity + homogeneity (or prove both separ.):

$$\begin{aligned} T(\lambda A + C) &= (\lambda A + C)B - B(\lambda A + C) \\ &= \lambda AB + CB - \lambda BA + -BC \\ &= \lambda(AB - BA) + (CB - BC) \\ &= \lambda T(A) + T(C) \end{aligned}$$

So T is a linear map.

b)
$$\begin{aligned} T\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix} - \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} \\ &= \begin{pmatrix} 3b-2c & 2a+3b-2d \\ 3d-3c-3a & 2c-3b \end{pmatrix} \end{aligned}$$

$$M(T) = \begin{array}{c} \\ a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 3 & -2 & 0 \\ 2 & 3 & 0 & -2 \\ -3 & 0 & -3 & 3 \\ 0 & -3 & 2 & 0 \end{pmatrix} \end{array}$$

← find by doing this from $T\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or by apply T to each of the standard basis vectors for M_2.

c) $\mathrm{rref}(M(T))$ is $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

notice $\mathrm{col}(M(T)) = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 2 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ -3 \end{pmatrix}\right\} \Rightarrow$ basis for $\mathrm{Im}(T) = \left\{\begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & -3 \end{pmatrix}\right\}$

and $\mathrm{null}(M(T)) = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s+t \\ 2/3s \\ s \\ t \end{pmatrix} \;\middle|\; s,t \in \mathbb{R}\right\}$ basis for $\mathrm{null}(M(T)) = \left\{\begin{pmatrix} -3 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

So basis for $\ker(T) = \left\{\begin{pmatrix} -3 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

d) Not injective → for any n and any B, $T(I) = IB - BI = B - B = 0$ where I is the $n \times n$ identity matrix. So $I_n \in \ker(T) \Rightarrow \ker(T) \neq \{0\}$ so T is not injective.
Not surjective → Since $I_n \in \ker(T)$ there is at least one free variable. $M(T)$ is $n^2 \times n^2$. So there's at most n^2-1 pivots $\Rightarrow \dim(\mathrm{im}(T)) \leq n^2-1$ but $\dim(M_{n\times n}) = n^2$

③ $M(S) = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$ $\quad M(T) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\quad M(T \circ S) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 2 & -1 & 3 \end{pmatrix}$$

④ a) Assume S and T are both injective. Is $S+T$ injective?

No → Let $S = I$, $T = -I$ and $U = V = \mathbb{R}^n$.

Then S, T both injective but $S+T = I + -I = 0$ not injective

Use same counterexample for both surjective.

b) Assume S is injective and $a \neq 0$. We want to show aS is injective.

Say aS not injective (for the sake of contradiction).

Then $\exists\, u \in U$ such that $(aS)(u) = 0$ and $u \neq 0$.

By definition, $\underbrace{(aS)u}_{= 0} = a(Su)$ and since $a \neq 0$, $Su = 0$.

But S is injective $\Rightarrow u = 0$ →←

Then aS must be injective.

Now, Suppose S is surjective. Then $\forall v \in V\ \exists\, u \in U$ s.t. $S(u) = v$.

We want to prove aS is surjective. Let $v \in V$. We need to find a $u \in U$ s.t. $aS(u) = v$. We know $\exists\, u'$ s.t. $S(u') = v$ and $(aS)(u') = a(Su')$
$= av$

Then $aS(\frac{1}{a}u') = a(S\frac{1}{a}u') = a\frac{1}{a}(Su') = v$ since S is linear. ☑

⑤ a) $T: P_2 \to P_2$ defined by $T(p) = p' - p$. Use $\mathcal{B} = \{1, x, x^2\}$

Then $[T]_{\mathcal{B}\to\mathcal{B}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$ Since this is upper triangular, Rank $= 3$.
$\boxed{\text{Then } T \text{ is both injective \& surjective}}$
So $\operatorname{Im}(T) = P_2$ and $\ker(T) = \{0\}$

b) $T: P_2 \to P_3$ defined by $T(p) = xp$ use $\mathcal{B}_1 = \{1, x, x^2\}$ & $\mathcal{B}_2 = \{1, x, x^2, x^3\}$

Then $[T]_{\mathcal{B}_1\to\mathcal{B}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Rank $= 3$, Nullity $= 0$

$\ker T = \{0\} \to \boxed{T \text{ is injective}}$

$\operatorname{Im} T = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}_2}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{B}_2}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{\mathcal{B}_2} \right\} = \operatorname{span}\{x, x^2, x^3\}$

$3 = \text{Rank} < \dim(P_3) = 4$ $\boxed{\text{So } T \text{ is not surjective.}}$

c) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x) = (x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3)$

$M(T) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ rank $= 3$, Nullity $= 0$

$\ker T = \{0\} \to \boxed{T \text{ is injective}}$

Rank $= \dim(\mathbb{R}^3) \to \boxed{T \text{ is surjective}}$ & $\operatorname{Im} T = \mathbb{R}^3$

d) $T: P_2 \to \mathbb{R}^2$ defined by $T(p(x)) = (p(0), p'(0))$ $\quad \mathcal{B}_1 = \{1, x, x^2\}$ $\quad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

Then $[T]_{\mathcal{B}_1\to\mathcal{B}_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ Rank $= 2 = \dim \mathbb{R}^2 \to \boxed{T \text{ is surjective}}$ and $\operatorname{Im} T = \mathbb{R}^2$

Nullity $= 1$. $\ker(T) = \operatorname{span}\{x^2\} \to \boxed{T \text{ is not injective}}$

free variable