Warm Up 4/25

① What are coordinates of a vector?

↳ column vector of coefficients corresponding to v when written as a linear combination of basis vectors.

② What is a linear map?

satisfies property of homogeneity ↓

↳ function that is additive $(T(u+v) = Tu + Tv)$ and homogeneous? $(T(\lambda v) = \lambda T v)$

How do we represent linear maps?

↳ explicitly, tell where basis is sent, with a matrix

How do we write the matrix of a linear map $T: V \to W$ with $\mathcal{B}_1, \mathcal{B}_2$ bases for V, W respectively?

① Find $T(v)$ $\forall$ $v \in \mathcal{B}_1$

② Find coordinates of $T(v)$ wrt $\mathcal{B}_2$

③ Columns of T = coordinates from ②

③ What is the fundamental theorem of linear maps?

If V finite dim'l & $T \in \mathcal{L}(V, W)$ Then $\dim(\text{range}(T)) + \dim(\ker(T)) = \dim V$

④ What does it mean if two vectors are orthogonal?

↳ $\langle u, v \rangle = 0$

Def: A set of vectors $S \subset V$ is orthonormal if S is orthogonal and every vector in S has norm 1.

Ex: Is $\mathcal{B} = \{1, x, \frac{1}{2}(3x^2-1)\}$ an orthonormal basis for P_2? (with $\langle , \rangle$ above)

Solution: We already observed $\langle 1, x \rangle = 0$ (And for now we assume Lin. Indep)

- $\langle 1, \frac{1}{2}(3x^2-1)\rangle = \int_{-1}^{1} \frac{1}{2}(3x^2-1)\,dx = \frac{1}{2}(x^3-x)\Big|_{-1}^{1} = \frac{1}{2}(1-1-(-1+1)) = 0.$
- $\langle x, \frac{1}{2}(3x^2-1)\rangle = \int_{-1}^{1} \frac{1}{2}(3x^2-1)x\,dx = \frac{1}{2}(\frac{3}{4}x^4 - \frac{1}{2}x^2)\Big|_{-1}^{1} = \frac{1}{8} - \frac{1}{8} = 0$

So $\mathcal{B}$ is orthogonal. We need to check norms:

- $\sqrt{\langle 1, 1\rangle} = \sqrt{\int_{-1}^{1} 1\,dx} = \sqrt{x\Big|_{-1}^{1}} = \sqrt{1+1} = \sqrt{2} \neq 1$ $(\|1\|)$
- $\sqrt{\langle x, x\rangle} = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\frac{1}{3}x^3\Big|_{-1}^{1}} = \sqrt{\frac{1}{3}+\frac{1}{3}} = \sqrt{\frac{2}{3}} \neq 1$ $(\|x\|)$
- $\sqrt{\langle \frac{1}{2}(3x^2-1), \frac{1}{2}(3x^2-1)\rangle} = \sqrt{\int_{-1}^{1} \frac{1}{4}(3x^2-1)^2\,dx} = \sqrt{\int_{-1}^{1} \frac{1}{4}(9x^4-6x^2+1)\,dx} = \sqrt{\frac{1}{4}(\frac{9}{5}x^5-2x^3+x)\Big|_{-1}^{1}} = \underset{\downarrow}{\overset{\|\frac{1}{2}(3x^2-1)\|}{\sqrt{\frac{2}{5}}}}$

Then $\mathcal{B}$ is not orthonormal.

Ex: $V = P_2$ with $\langle p, q\rangle = p_0q_0 + p_1q_1 + p_2q_2$ where $p = p_0 + p_1x + p_2x^2$, $q = q_0 + q_1x + q_2x^2$

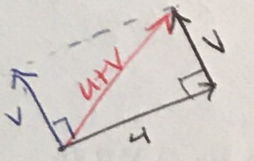
On your own → verify this is inner product and check that $\mathcal{B} = \{1, x, x^2\}$ is orthonormal (Note: $\langle , \rangle$ same as $\mathbb{R}^3$ & $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$ O.N.)

$\boxed{\text{RESULTS}}$

Pythagorean Theorem: If u and v are orthogonal, $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Proof:
$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle$$
$$= \langle u+v, u\rangle + \langle u+v, v\rangle = \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle$$
$$= \|u\|^2 + 0 + 0 + \|v\|^2 \quad //$$

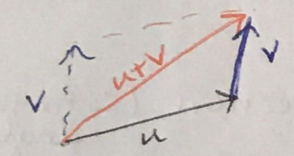
In $\mathbb{R}^2$:



Cauchy-Schwarz Inequality: Suppose $u, v \in V$. Then $|\langle u,v\rangle| \leq \|u\| \cdot \|v\|$. Equality holds iff $u = cv$. (Proof in Axler)

Triangle Inequality: Suppose $u, v \in V$. Then $\|u+v\| \leq \|u\| + \|v\|$. Equality holds iff $u = cv$ where $c \geq 0$. (Proof in Axler, but try on your own)

In $\mathbb{R}^2$:



Thm: Let V be an inner product space. If $S = \{v_1, \dots, v_m\}$ is a set of orthogonal vectors with $0 \notin S$, then S is linearly independent.

Proof: Assume, for the sake of contradiction, that S is linearly dependent.

Then $a_1v_1 + \dots + a_mv_m = 0$ where not all a_i's are 0.

Assume $a_j \neq 0$. We know $\langle 0, v_j\rangle = 0$ so $\langle a_1v_1 + \dots + a_jv_j + \dots + a_mv_m, v_j\rangle = 0$

Then $a_1\langle v_1, v_j\rangle + \dots + a_j\langle v_j, v_j\rangle + \dots + a_m\langle v_m, v_j\rangle = 0$

Since S is orthogonal, $\langle v_i, v_j \rangle = 0$ $\forall$ $i \neq j$, leaving $a_j \langle v_j, v_j \rangle = 0$.
We've assumed $a_j \neq 0$ so $\langle v_j, v_j \rangle = 0$ and since $\langle \, , \, \rangle$ is positive definite, this implies $v_j = 0$, which contradicts $0 \notin S$. $\rightarrow\leftarrow$
Therefore, $a_j = 0$ and so S must be linearly independent. $\boxtimes$

Corollary: If V is an inner product space of dimension n & $\mathcal{B} = \{v_1, \dots, v_n\}$ is a set of nonzero orthogonal vectors, then $\mathcal{B}$ is a basis for V.

And so we have a new way to write any vector in V:

Theorem: Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be an orthogonal basis for V. Then for any $v \in V$

$$v = \underbrace{\frac{\langle v_1, v \rangle}{\langle v_1, v_1 \rangle}} v_1 + \frac{\langle v_2, v \rangle}{\langle v_2, v_2 \rangle} v_2 + \dots + \frac{\langle v_n, v \rangle}{\langle v_n, v_n \rangle} v_n.$$

$\uparrow$ $\uparrow$ $\uparrow$

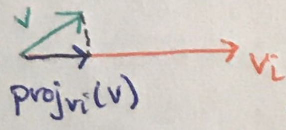
coordinates with respect to $\mathcal{B}$.

Proof: Since $\mathcal{B}$ is a basis for V, $\exists$ $c_1, \dots, c_n$ s.t. $v = c_1 v_1 + \dots + c_n v_n$
Then for any i, $\langle v_i, v \rangle = \langle v, v_i \rangle = \langle c_1 v_1 + \dots + c_n v_n, v_i \rangle$

$$= c_1 \langle v_1, v_i \rangle + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle$$

Since $\mathcal{B}$ is orthogonal, we have $\langle v_i, v_i \rangle = c_i \langle v_i, v_i \rangle$ and so $c_i = \dfrac{\langle v_i, v \rangle}{\langle v_i, v_i \rangle}$ $\boxtimes$

Think of $\dfrac{\langle v_i, v \rangle}{\langle v_i, v_i \rangle} v_i$ as the part of v in the v_i direction.



projection of v onto v_i

Note, if $\mathcal{B}$ were orthonormal, then $\langle v_i, v_i \rangle = 1$ $\forall i$ and we'd have

$$v = \langle v_1, v \rangle v_1 + \dots + \langle v_n, v \rangle v_n.$$

To construct an orthonormal basis $\{e_1, \dots, e_n\}$ from any basis $\{v_1, \dots, v_n\}$ we have the following:

Gram-Schmidt Procedure

① $e_1 = \dfrac{v_1}{\|v_1\|}$ (unit vector in direction of v_1)

② $$e_2 = \frac{v_2 - \overbrace{\langle v_2, e_1 \rangle e_1}}{\|v_2 - \langle v_2, e_1 \rangle e_1\|}$$
$\rightarrow$ piece of v_2 that was in e_1 direction (so fix the angle)
$\leftarrow$ cut down to norm 1.

③ $$e_3 = \frac{v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2}{\|v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2\|}$$

$\vdots$

ⓚ $1 \leq k \leq n$ $$e_k = \frac{v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i}{\|v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i\|}$$

NOTE: This is ugly to do by hand. Thank goodness for computers.