ⓐ $\forall v \in V$, $\langle v,v\rangle \geq 0$ and $\langle v,v\rangle = 0$ if and only if $v=0$ ($\langle , \rangle$ is positive definite)
ⓑ $\forall u,v,w \in V$ and $a \in \mathbb{R}$, $\langle au+v, w\rangle = a\langle u,w\rangle + \langle v,w\rangle$ ($\langle , \rangle$ is linear in first variable)
ⓒ $\forall u,v \in V$, $\langle u,v\rangle = \langle v,u\rangle$ (if $\mathbb{C}$, $\langle u,v\rangle = \overline{\langle v,u\rangle}$) ($\langle , \rangle$ is symmetric)

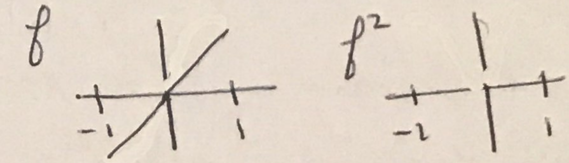
Def: A vector space V with an inner product is called an inner product space.

Recall: For $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\langle x,x\rangle} = \sqrt{x \cdot x}$

Def: If V is an inner product space, the norm of any $v \in V$ is defined by $\|v\| = \sqrt{\langle v,v\rangle}$

Ex:
- $\mathbb{R}^n$ is an inner product space with $\langle , \rangle$ = dot product
- $\mathbb{R}^n$ is a different inner product space with a weighted dot product:
  $\langle x,y\rangle = c_1x_1y_1 + c_2x_2y_2 + \dots + c_nx_ny_n$ where $c_i \in \mathbb{R}_{>0}$
  $\|x\| = \sqrt{c_1x_1^2 + c_2x_2^2 + \dots + c_nx_n^2}$
- $V = \{f : [-1,1] \to \mathbb{R} \mid f$ is continuous on $[-1,1]\}$ is an inner product space with $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

Check: ⓐ $\langle f,f\rangle = \int_{-1}^{1} (f(x))^2\,dx$ $\quad (f(x))^2 \geq 0$ so area underneath from -1 to 1 is ≥ 0!



ⓑ $\langle af+g, h\rangle$ for $a \in \mathbb{R}$, $f,g,h \in V$

$= \int_{-1}^{1} (af+g)(x)\,h(x)\,dx$

$= \int_{-1}^{1} (af(x)+g(x))\,h(x)\,dx$

$= a\int_{-1}^{1} f(x)h(x)\,dx + \int_{-1}^{1} g(x)h(x)\,dx = a\langle f,h\rangle + \langle g,h\rangle$

ⓒ $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} g(x)f(x)\,dx = \langle g,f\rangle$

So $\|f\| = \sqrt{\int_{-1}^{1} (f(x))^2\,dx}$

On your own: $P(\mathbb{R})$ is an inner product with $\langle p,q\rangle = \int_0^\infty p(x)\,q(x)e^{-x}\,dx$

Properties:
- $\ast$ $\langle 0,u\rangle = 0$ for every $u \in V$ (Prove on your own)
- $\ast$ $\langle u,0\rangle = 0$ for every $u \in V$ Proof: $\langle u,0\rangle = \langle 0,u\rangle$ by ⓒ & then use above ☒
- $\ast$ $\|v\| = 0$ if and only if $v = 0$. Proof: $\|v\| = \sqrt{\langle v,v\rangle}$ and $\langle v,v\rangle = 0$ if and only if $v=0$ ☒
- $\ast$ $\|\lambda v\| = |\lambda|\,\|v\|$ $\forall \lambda \in \mathbb{R}$ Proof: $\|\lambda v\| = \sqrt{\langle \lambda v, \lambda v\rangle} = \sqrt{\lambda\langle v, \lambda v\rangle} = \sqrt{\lambda\langle \lambda v, v\rangle} = \sqrt{\lambda^2\langle v,v\rangle} = |\lambda|\,\|v\|$ ☒

Def: Two vectors u and v are called orthogonal if $\langle u,v\rangle = 0$ Note: like perpendicular

Ex: standard basis vectors in $\mathbb{R}^n$ are all orthogonal $\rightarrow \langle (1,0,0),(0,1,0)\rangle = 0$
- $V = P_2$, $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$.
  $\langle 1,x\rangle = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$ So 1 and x are orthogonal.

Def: A set of vectors $S \subset V$ is said to be orthogonal if every pair of vectors from S is orthog.

Warm up 4/22

① Is a subset of a known vector space always a vector space?

No! must have 0, closed under $\cdot$, and closed under + (called subspace)

② What is the span of a set?

↳ set of all linear combinations of vectors in the set

Is $\operatorname{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$ a subspace of $\mathbb{R}^3$?

yes! span of a set of vectors is smallest subspace containing those vectors.

③ When is a vector space finite dimensional?

↳ If it has a finite spanning set

What do we know about dimensions of subspaces?

↳ always less than or equal to dimension of the larger vector space

4 What is the column space of a matrix? Null space?

Column space is the vector space spanned by the columns of the matrix.
Null space of A is the set of a x s.t. $Ax = 0$.
rank(A) + nullity(A) = # columns of A