Review Questions to warm up:

① What's a basis?

↳ linearly independent spanning set for a vector space V

ⓐ What does it mean for a set to be linearly independent? ($S=\{v_1, \ldots, v_n\}$)

↳ If the only solution to $a_1v_1+\ldots+a_mv_m=0$ is $a_1=\ldots=a_m=0$ (trivial).

ⓑ What's a spanning set?

↳ a set of vectors whose span (set of all linear combinations) is V.

② What is the dimension of a vector space V?

↳ size of any basis of V (constant!)

How does knowing the dimension of a vector space help us with finding a basis?

↳ any collection of n linearly independent vectors is a basis if $\dim V=n$

↳ any spanning set of size n is a basis if $\dim V=n$.

③ What's an eigenvalue?

↳ For $T\in\mathcal{L}(V)$ (or $A\in M_{n\times n}$) λ an eigenvalue if $\exists\, v\neq 0$ s.t. $T(v)=\lambda v=Av$

What's an eigenvector?

↳ nonzero vector v associated with λ

How do you find eigenvalues?

↳ roots of the characteristic polynomial

What do we mean by "multiplicity" of an eigenvalue?

↳ algebraic multiplicity = # times eigenvalue appears as a root.

geometric multiplicity = dimension of corresponding eigenspace

④ How do we multiply matrices?

↳ dot product of row i of A & column j of B $=(AB)_{ij}$

⑤ Name as many vector spaces isomorphic to $\mathbb{R}^4$ as possible.

We had in (4) that if A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent

Thm: Let A be an nxn matrix. A is diagonalizable if and only if A has n linearly independent eigenvectors

OR, in terms of linear operators:

Let V be a finite-dimensional vector space ($\dim V = n$) and $T: V \to V$ a linear map

T is diagonalizable if and only if there exists a basis of V, all of whose vectors are eigenvectors of T. $B = \{v_1, \ldots, v_n\}$

In that case,

$M(T) = A = PDP^{-1}$ where $P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

where λ_i is the eigenvalue for eigenvector v_i

Additionally, A is diagonalizable if algebraic multiplicity always equals geometric multiplicity.

Why?

Ex $A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ characteristic polynomial of A is $(2-\lambda)^2(1-\lambda)$

$\lambda = 2$ has alg. mult. 2; $\lambda = 1$

Eigenspaces: $\underline{\lambda = 1}$: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix}$ so $E_1 = \text{span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}$

$\underline{\lambda = 2}$: $\begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_2 = 0$ so $E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

geom. mult. 2

Then even though there are not 3 distinct eigenvalues, there are 3 linearly independent eigenvectors. Then we can choose 3 as a basis for $\mathbb{R}^3$

$\Rightarrow$ A is diagonalizable with $P = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (check!)

Non Ex: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ characteristic polynomial is $(2-\lambda)^2(1-\lambda)$

$\lambda = 2$ has alg. mult 2; $\lambda = 1$

E_1: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_1 = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

E_2: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \Rightarrow \dim(E_2) = 1$ geom. mult 1 $\rightarrow$ not enough eigenvectors to build a basis for $\mathbb{R}^3$

$\therefore$ A is not diagonalizable.

Inner Product Spaces → Axler 6A+6B

Isomorphisms allow us to treat other, abstract vector spaces like $\mathbb{R}^n$. $\mathbb{R}^n$ also uses dot product to think about lengths + angles btwn vectors

Recall: $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i = \langle x, y \rangle$ ← possibly new notation

Def Let V be a vector space over $\mathbb{R}$ (for simplicity). An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ (input is two vectors & output is real #) so that

ⓐ $\forall v \in V$, $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$ ($\langle , \rangle$ is positive definite)

ⓑ $\forall u, v, w \in V$ and $a \in \mathbb{R}$, $\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$ ($\langle , \rangle$ is linear in first variable)

ⓒ $\forall u, v \in V$, $\langle u, v \rangle = \langle v, u \rangle$ (if $\mathbb{C}$, $\langle u, v \rangle = \overline{\langle v, u \rangle}$) ($\langle , \rangle$ is symmetric)