Facts about Eigenvalues & Eigenvectors $A \in M_{n \times n}$

① If A is triangular or diagonal, the eigenvalues are the diagonal entries

② $\det(A) =$ product of eigenvalues of A with multiplicity

③ $\mathrm{trace}(A) =$ sum of eigenvalues of A with multiplicity
↳ sum of diagonal entries of A

④ If A has n distinct eigenvalues, then the eigenvectors are linearly independent

⑤ Suppose A is invertible. Then

a) 0 is not an eigenvalue.
→ Because by ②, $\det(A) =$ product of eigenvalues and if A is invertible then $\det(A) \neq 0$.

b) If λ is an eigenvalue for A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}
→ Because $Av = \lambda v$ for some $v \neq 0$
So $A^{-1}(Av) = A^{-1}(\lambda v) \Rightarrow v = \lambda A^{-1} v \Rightarrow A^{-1} v = \frac{1}{\lambda} v$

⑥ If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.
→ Because $Av = \lambda v$ So
$$\begin{aligned} A^k v &= A^{k-1}(Av) \\ &= A^{k-1}(\lambda v) = \lambda (A^{k-2}(Av)) \\ &= \lambda^2 (A^{k-2} v) \ldots = \lambda^{k-1}(Av) \\ &= \lambda^k v. \end{aligned}$$

⑦ If A is similar to B ($A = PBP^{-1}$) and λ is an eigenvalue for A with eigenvector v, then λ is an eigenvalue for B with eigenvector $P^{-1}v$
→ Because $Av = \lambda v \Rightarrow (PBP^{-1})v = \lambda v$
$\Rightarrow B(P^{-1}v) = P^{-1}(\lambda v) = \lambda(P^{-1}v)$

Prop: Similar matrices have equal characteristic polynomials.

Proof. Suppose A and B are similar matrices $\Rightarrow A = PBP^{-1}$. for some inv. P

Then the characteristic polynomial of A is
$$\begin{aligned} \det(A - \lambda I) &= \det(PBP^{-1} - \lambda I) \\ &= \det(PBP^{-1} - \lambda P I P^{-1}) \quad \text{(Why?)} \\ &= \det(P(B - \lambda I)P^{-1}) \\ &= \det(P) \det(B - \lambda I) \det(P^{-1}) \quad \det(P^{-1}) = \frac{1}{\det(P)} \\ &= \det(B - \lambda I) \text{ which is the char. poly. of } B \quad \square \end{aligned}$$

* #s so mult. is commut.

Def: An $n \times n$ matrix is diagonalizable if it is similar to a diagonal matrix. $\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$

Ex: $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$ check that A is diagonalizable with $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$.

Solution: $P^{-1} = \frac{1}{-2+1}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$ A similar to $D \rightarrow A = PDP^{-1}$
then $D = P^{-1}AP$

$D = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

$\begin{pmatrix} 10 & 5 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ Diagonal ✓

- Then $\det(A - \lambda I) = \det(D - \lambda I) = (5-\lambda)(3-\lambda)$ (Easier to find)
- Eigenvalues of A are eigenvalues of $D \to \lambda_1 = 5, \lambda_2 = 3$
- $\det(A) = \det(D) = 15$ → both A + D are invertible!
- $\text{trace}(A) = \text{trace}(D) = 8$ ✓ (can check)

Let's check eigenvalues of A^k:

$\forall k > 0$; $A^k = (PDP^{-1})^k$

$= \underbrace{(PDP^{-1})(PDP^{-1}) \dots (PDP^{-1})(PDP^{-1})}_{k \text{ times}}$

What happens?

$= PD^kP^{-1}$

Note: $\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}$

$= P \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^k P^{-1}$

$= P \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix} P^{-1}$ ← SO: A^k is similar to a diagonal matrix! ⇒ eigenvalues of A^k are $5^k, 3^k$ ✓

$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 5^k & 3^k \\ -5^k & -2\cdot 3^k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ -2\cdot 5^k + 2\cdot 3^k & -5^k + 2\cdot 3^k \end{pmatrix}$ ← Now it's easy to find any power of k

Negative Exponent? $A^{-k} = (PDP^{-1})^{-k}$

$= ((PDP^{-1})^{-1})^k$

$= ((DP^{-1})^{-1}P^{-1})^k$ socks-shoes

$= (PD^{-1}P^{-1})^k$

$= P(D^{-1})^k P^{-1}$ Easy to find!

$= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}^k & 0 \\ 0 & \frac{1}{3}^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ Same as above with $-k$

We had in ④ that if A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent.

Thm Let A be an $n \times n$ matrix. A is diagonalizable if and only if A has n linearly independent vectors

OR, in terms of linear operators:

Let V be a finite-dimensional vector space ($\dim V = n$) and $T: V \to V$ a linear map. T is diagonalizable if and only if there exists a basis of V, all of whose vectors are eigenvectors of T. $B = \{v_1, \dots, v_n\}$