Warm Up & Review

Let $V=\mathbb{R}^2$, $\mathcal{B}_1=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\mathcal{B}_2=\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}$.

Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear map defined by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+3y\\3x+y\end{pmatrix}$.

a) Find T with respect to $\mathcal{B}_1,\mathcal{B}_1$

b) Find T with respect to $\mathcal{B}_2,\mathcal{B}_2$

c) Find the COB matrix <u>from</u> $\mathcal{B}_2$ <u>to</u> $\mathcal{B}_1$ (Hint: this is the easy one). Call it P.

d) Find P^{-1} (COB matrix <u>from</u> $\mathcal{B}_1$ <u>to</u> $\mathcal{B}_2$)

e) Find $P[T]_{\mathcal{B}_1,\mathcal{B}_1}P^{-1}$. What do you get?

Solutions: a)

$$[T]_{\mathcal{B}_1,\mathcal{B}_1}=\begin{pmatrix}1&3\\3&1\end{pmatrix}$$

b) $$[T]_{\mathcal{B}_2,\mathcal{B}_2}=\left(T(1,1)_{\mathcal{B}_2}\;\middle|\;T(1,-1)_{\mathcal{B}_2}\right)=\begin{pmatrix}4&0\\0&-2\end{pmatrix}$$

$$T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix}+0\begin{pmatrix}1\\-1\end{pmatrix}$$
$$T\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-2\\2\end{pmatrix}=-2\begin{pmatrix}1\\-1\end{pmatrix}=0\begin{pmatrix}1\\1\end{pmatrix}-2\begin{pmatrix}1\\-1\end{pmatrix}$$

c) $$P=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

d) $$P^{-1}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}$$

e) $$P[T]_{\mathcal{B}_1,\mathcal{B}_1}P^{-1}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&3\\3&1\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}$$
$$=\begin{pmatrix}4&4\\-2&2\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}=\begin{pmatrix}4&0\\0&-2\end{pmatrix}=[T]_{\mathcal{B}_2,\mathcal{B}_2}!$$

Def: Let A,B be two $n\times n$ matrices. A is <u>similar</u> to B if $A=PBP^{-1}$ for some invertible $n\times n$ matrix P.

→ **Note from Ex:** If $T:V\to V$ is a linear operator & $\mathcal{B}_1,\mathcal{B}_2$ are two bases for V, then $[T]_{\mathcal{B}_1,\mathcal{B}_1}$ is similar to $[T]_{\mathcal{B}_2,\mathcal{B}_2}$ via the COB matrix, P.

Facts

* A is similar to itself (How? → $A=I_nAI_n^{-1}$). *reflexive property*
* If A is similar to B, B is similar to A. (How? → $A=PBP^{-1}$ so $B=P^{-1}AP=(P^{-1})A(P^{-1})^{-1}$). *symmetric property*
* If A is similar to B and B is similar to C, then A is similar to C. (How? $A=PBP^{-1}$ $B=QCQ^{-1}$
$A=P(QCQ^{-1})P^{-1}=(PQ)C(QP)^{-1}$). *transitive property*

∴ Similarity of matrices = equivalence relation → can be broken into equivalence classes

Facts about Eigenvalues & Eigenvectors $\quad A \in M_{n \times n}$

① If A is triangular or diagonal, the eigenvalues are the diagonal entries.

② $\det(A) =$ product of eigenvalues of A with multiplicity

③ $\operatorname{trace}(A) =$ sum of eigenvalues of A with multiplicity

↳ sum of diagonal entries of A

④ If A has n distinct eigenvalues, then the eigenvectors are linearly independent

⑤ Suppose A is invertible. Then

a) 0 is not an eigenvalue.

→ Because by ②, $\det(A) =$ product of eigenvalues and if A is invertible then $\det(A) \neq 0$.

b) If λ is an eigenvalue for A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

→ Because $Av = \lambda v$ for some $v \neq 0$

So $A^{-1}(Av) = A^{-1}(\lambda v) \Rightarrow v = \lambda A^{-1} v \Rightarrow A^{-1} v = \frac{1}{\lambda} v$.

⑥ If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

→ Because $Av = \lambda v$ So $A^k v = A^{k-1}(Av)$

$$= A^{k-1}(\lambda v) = \lambda (A^{k-2}(Av))$$

$$= \lambda^2 (A^{k-2} v) \ldots = \lambda^{k-1}(Av)$$

$$= \lambda^k v.$$

⑦ If A is similar to B ($A = PBP^{-1}$) and λ is an eigenvalue for A with eigenvector v, then λ is an eigenvalue for B with eigenvector $P^{-1}v$

→ Because $Av = \lambda v \Rightarrow (PBP^{-1})v = \lambda v$

$$\Rightarrow B(P^{-1}v) = P^{-1}(\lambda v) = \lambda(P^{-1}v)$$

Prop: Similar matrices have equal characteristic polynomials.

Proof: Suppose A and B are similar matrices $\Rightarrow A = PBP^{-1}$. for some inv. P

Then the characteristic polynomial of A is

$$\det(A - \lambda I) = \det(PBP^{-1} - \lambda I)$$

$$= \det(PBP^{-1} - \lambda P I P^{-1}) \quad \text{(Why?)}$$

$$= \det(P(B - \lambda I)P^{-1})$$

$$= \det(P)\det(B - \lambda I)\det(P^{-1}) \quad \det(P^{-1}) = \frac{1}{\det(P)} \quad \text{* \#s so mult. is commut.}$$

$$= \det(B - \lambda I) \text{ which is the char. poly. of } B \ \boxtimes$$

Def: An $n \times n$ matrix is diagonalizable if it is similar to a diagonal matrix. $\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$