Ex:

$$\det\begin{pmatrix}1&3&2\\0&4&5\\-1&2&7\end{pmatrix} = 0(-1)^{2+1}\begin{vmatrix}3&2\\2&7\end{vmatrix} + 4(-1)^{2+2}\begin{vmatrix}1&2\\-1&7\end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix}1&3\\-1&2\end{vmatrix}$$

$$\downarrow R_1+R_3 \qquad = 0 + 4(9) - 5(5) = 36-25 = \boxed{11}$$

$$\det\begin{pmatrix}1&3&2\\0&4&5\\0&5&9\end{pmatrix} = 0(-1)^{2+1}\begin{vmatrix}3&2\\5&9\end{vmatrix} + 4(-1)^{2+2}\begin{vmatrix}1&2\\0&9\end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix}1&3\\0&5\end{vmatrix}$$

$$= 0 + 4(9) - 5(5) = \boxed{11}$$

Def: A function f of the rows of a matrix A is called multilinear if f is a linear function in each row when the others are fixed.

i.e. $f(a_1, \dots, ka_\ell + k'a_\ell', \dots, a_n) = k f(a_1, \dots, a_\ell, \dots, a_n) + k' f(a_1, \dots, a_\ell', \dots, a_n)$

Ex: Determinant (only alternating ④ one!)

Upshot of ③, ④, ⑤: Row reducing changes determinant to nonzero multiple of $\det(A)$

So, if A invertible, $\mathrm{rref}(A) = I_n \Rightarrow \det(A)$ can be anything but 0!

Add to $\Leftrightarrow$ Thm: A invertible $\Leftrightarrow \det(A) \neq 0$.

⑥ $\det(A^T) = \det(A)$

⑦ $\det(AB) = \det(A)\det(B)$

⑧ $\det(A^m) = (\det(A))^m \quad \forall m \in \mathbb{Z}_{>0}$

⑨ $\det(A^{-1}) = (\det(A))^{-1} = \frac{1}{\det(A)}$ if A is invertible.

In Summary,

☆ A is invertible iff $\mathrm{rref}(A) = I_n$

$\cap$ iff $\mathrm{rank}(A) = n$

$M_{n\times n}$ iff $\mathrm{null}(A) = \{0\}$

iff $\mathrm{col}(A) = \mathbb{R}^n$

iff columns of A are linearly independent

iff columns of A span $\mathbb{R}^n$

iff columns of A are a basis for $\mathbb{R}^n$

☆ iff $Ax = 0$ has only one solution, $x = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$

☆ iff $\det(A) \neq 0$.

Back to eigenvalues:

λ is an eigenvalue for A if $\exists$ nonzero v s.t. $Av = \lambda v$, or $(A-\lambda I)v = 0$.

So: goal is to find λs where $(A-\lambda I)v = 0$ has nontrivial solutions ☆

☆ we find this by finding when $A - \lambda I_n$ is not invertible

☆ so, find λs so that $\det(A - \lambda I_n) = 0$

Remarks: * $\det(A - \lambda I_n)$ is the characteristic polynomial for A.

* $\det(A - \lambda I_n) = 0$ is the characteristic equation for A.

* $\deg(\det(A - \lambda I_n)) = n$, so A has n eigenvalues counting multiplicity
?

Ex: Find all eigenvalues and eigenspaces for $A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$

Solution: By our prior thought process, eigenvalues occur where $\det(A-\lambda I_n)=0$

$$\det(A-\lambda I_n) = \det\begin{pmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{pmatrix} = (2-\lambda)(-5-\lambda)+12 = -10+5\lambda-2\lambda+\lambda^2+12$$
$$= \underbrace{\lambda^2+3\lambda+2}_{\text{characteristic polynomial}}$$

$\det(A-\lambda I_n)=0 = \lambda^2+3\lambda+2 \leftarrow$ characteristic equation

$0=(\lambda+1)(\lambda+2)$ $\qquad \lambda_1=-1,\ \lambda_2=-2$

$\lambda_1=-1$ Plug back in to $A-\lambda I_n \rightarrow$ want $v\neq 0$ s.t. $(A-\lambda I_n)v=0$

$$\begin{pmatrix} 3 & -12 \\ 1 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix} \quad v_1-4v_2=0 \quad v=\begin{pmatrix} 4s \\ s \end{pmatrix} \quad \text{So } E_{-1} = \text{span}\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}$$

$E_{-1} = \{v\in\mathbb{R}^2 \mid Av=-1v\}$

-1-eigenspace

oyo: $\lambda_2=-2$ (Answer: $E_{-2}=\text{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$

Ex: Find eigenvalues and eigenspaces for $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$

Solution:

$$\det(A-\lambda I_n) = \det\begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 5 & -10 \\ 1 & 0 & 2-\lambda & 0 \\ 1 & 0 & 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(-1)^{1+1}\begin{vmatrix} 1-\lambda & 5 & -10 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} +0+0+0$$

upper triangular

$$= (1-\lambda)(1-\lambda)(2-\lambda)(3-\lambda) \leftarrow \text{already factored!}$$

$\underbrace{\lambda_1=1}_{\text{algebraic multiplicity 2.}}$, $\lambda_2=2$ $\lambda_3=3$ (from setting $=0$)

Eigenspace for $\lambda_1=1$ (Solve $(A-I)v=0$)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} v_1=-2v_4 \\ v_3=2v_4 \\ v_2 \text{ anything} \end{matrix} \quad v=\begin{pmatrix} -2t \\ s \\ 2t \\ t \end{pmatrix}. \quad \text{So } E_1 = \left\{\begin{pmatrix} -2t \\ s \\ 2t \\ t \end{pmatrix} \middle| s,t\in\mathbb{R}\right\}$$

$$= \text{Span}\left\{\begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

geometric multiplicity 2.

Fact: geometric multiplicity of $\lambda \leq$ algebraic multiplicity of λ.

Ex: $A=\begin{pmatrix} 0 & 5 \\ -5 & 10 \end{pmatrix}$ $\quad A-\lambda I = \begin{pmatrix} -\lambda & 5 \\ -5 & 10-\lambda \end{pmatrix}$

$$\det\begin{pmatrix} -\lambda & 5 \\ -5 & 10-\lambda \end{pmatrix} = -\lambda(10-\lambda)+25 = \lambda^2-10\lambda+25 = (\lambda-5)^2 \quad \lambda=5 \text{ has alg. mult. 2}$$

$$E_5: \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad v_1=v_2 \quad \text{So } E_5 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \quad \lambda=5 \text{ has geom. mult. 1}$$

Ex: $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad A - \lambda I = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix}$

$$\det(A-\lambda I) = -\lambda(-1)^{1+1} \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} + 0 + 0$$

$$= -\lambda(\lambda^2+1) = -\lambda^3 - \lambda = 0$$

complex conjugate pairs

$$-\lambda(\lambda^2+1) = 0 \qquad \lambda = 0,\ \lambda = i,\ \lambda = -i$$