To define determinant for any n we need:

this is $\downarrow$ (i,j) minor

Def: If A is an $n\times n$ matrix, the (i,j)-cofactor of A is $A_{ij} = (-1)^{i+j}|C_{ij}|$ where C_{ij} is an $(n-1)\times(n-1)$ matrix obtained from A by removing i^{th} row + j^{th} col.

Ex:

$$A = \begin{pmatrix} -8 & -7 & -6 & -5 \\ -4 & -3 & -2 & -1 \\ 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{pmatrix} \qquad A_{24} = (-1)^{2+4}\begin{vmatrix} -8 & -6 & -5 \\ -4 & -2 & -1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} -8 & -7 & -5 \\ 0 & 1 & 3 \\ 4 & 5 & 7 \end{vmatrix}$$

Back to our definition:

notice signs will alternate

Def: $\det(A) = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$ where A is $n\times n$ with $n \geq 2$. This is called the cofactor expansion along row 1.

Ex: Find $\det(A)$ where $A = \begin{pmatrix} -5 & 0 & 2 \\ 3 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}$

Solution: $\det(A) = -5(-1)^{1+1}\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + 0(-1)^{1+2}\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix}$

$= -5(1)(3) + 0 + 2(1)(1) = -15+2 = \boxed{-13}$

Prop: If A is an $n\times n$ matrix, $\det(A)$ can be found by doing a cofactor expansion along any row or column of A.

Above Ex: Expand along second column:

$\det(A) = 0(-1)^{1+2}\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} -5 & 2 \\ -1 & 3 \end{vmatrix} + 0(-1)^{3+2}\begin{vmatrix} -5 & 2 \\ 3 & -1 \end{vmatrix}$

$= 0 + (-15+2) + 0 = \boxed{-13}$ same ✓

Properties of Determinants

① If A has a row of zeros or a column of zeros, $\det(A) = 0$

② If A is triangular or diagonal, then $\det(A)$ is the product of elements along the main diagonal.

triangular: all zeros above or below diagonal; diagonal: both

Ex:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & -2 & 3 \end{pmatrix} \qquad \det A = 2(-1)^{1+1}\begin{vmatrix} 4 & 0 \\ -2 & 3 \end{vmatrix} + 0(-1)^{1+2}\begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} + 0(-1)^{1+3}\begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix}$$

$= 2(4(3)-0) + 0 + 0$

$= 2\cdot 4\cdot 3 = \boxed{24}$

Challenge: Prove this inductively

③ If a row or column of A is multiplied by a scalar k to get B, then $k\det A = \det B$

$\Rightarrow$ If we multiply A by k to get B, then $\det(B) = k^n\det(A)$

④ If two rows are switched in A to get B, then $\det(B) = -\det(A)$

⑤ If a multiple of one row (or column) is added to another to get B, then $\det(A) = \det(B)$

Ex

$$\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \\ -1 & 2 & 7 \end{pmatrix} = 0(-1)^{2+1}\begin{vmatrix} 3 & 2 \\ 2 & 7 \end{vmatrix} + 4(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ -1 & 7 \end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= 0 + 4(9) - 5(5) = 36-25 = \boxed{11}$$

$\downarrow R_1+R_3$

$$\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \\ 0 & 5 & 9 \end{pmatrix} = 0(-1)^{2+1}\begin{vmatrix} 3 & 2 \\ 5 & 9 \end{vmatrix} + 4(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 0 & 9 \end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix}$$

$$= 0 + 4(9) - 5(5) = \boxed{11}$$

Def: A function f of the rows of a matrix A is called multilinear if f is a linear function in each row when the others are fixed.

i.e. $f(a_1, \dots, ka_\ell + k'a_\ell', \dots, a_n) = k f(a_1, \dots, a_\ell, \dots, a_n) + k' f(a_1, \dots, a_\ell', \dots, a_n)$

Ex Determinant (only alternating ④ one!)

Upshot of ③,④,⑤: Row reducing changes determinant to nonzero multiple of $\det(A)$

So, if A invertible, $\mathrm{rref}(A) = I_n \Rightarrow \det(A)$ can be anything but 0!

Add to $\Leftrightarrow$ Thm: A invertible $\Leftrightarrow \det(A) \neq 0$.

⑥ $\det(A^T) = \det(A)$

⑦ $\det(AB) = \det(A)\det(B)$

⑧ $\det(A^m) = (\det(A))^m \quad \forall m \in \mathbb{Z}_{>0}$

⑨ $\det(A^{-1}) = (\det(A))^{-1} = \frac{1}{\det(A)}$ if A is invertible.