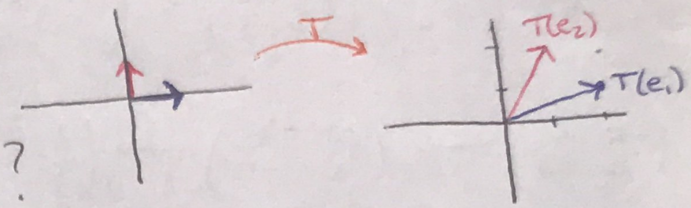
Our next topic restricts our view to only linear operators (and so, only square matrices)

Warm up Ex: Consider the linear mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

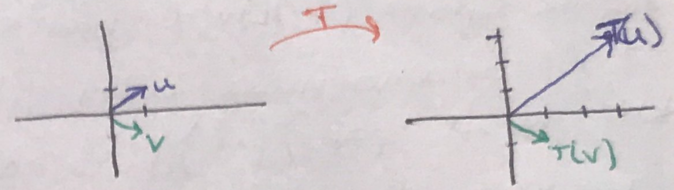
$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Notice what T does to the standard basis:



What does T do to $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

Solution: $Au = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3u$

$Av = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1)v$



Notice $\to$ T just stretches u and v ($Au = \ell u$ for some $\ell \in \mathbb{R}$)

Def: Suppose $T \in \mathcal{L}(V)$ (or A is the corresponding $n \times n$ matrix) a number $\lambda \in F$ is called an eigenvalue of T (or A) if $\exists\, v \in V$ s.t. $v \neq 0$ and $T(v) = \lambda v$. We call v an eigenvector of T (or A) corresponding to the eigenvalue λ. Together, (λ, v) is called an eigenpair

Above Ex: $(3, u)$ and $(1, v)$ are eigenpairs for T. (or A)

OR $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to 3 for T (or A).

On your own: Let $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$ $u = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ $v = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

Are u and v eigenvectors of A? If so, find the corresponding eigenvalue.

Solution: $Au = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} = -4u$ ✓ $(-4, u)$ is an eigenpair

$Av = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix} \times \neq \ell\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ v is not an eigenvector

You've actually seen this (in a way) before:

Exponential growth differential equation:

$$\frac{dy}{dx} = ky$$

If we consider $\{$all functions with derivatives of all orders$\} = V$, this is a vector space and so given equation says that y is an eigenvector of the linear mapping $T = \frac{dy}{dx}: V \to V$ with eigenvalue $\lambda = k$. Solving equation = finding eigenvectors ($y = ce^{kx}$)

Ex Show that 7 is an eigenvalue of $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$ (i.e. $Av = 7v$ for some $v \neq 0$)

Notice: $7v = 7\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} 7v_1 \\ 7v_2 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ (in general, $cv = cI_n v$ for $v \in \mathbb{R}^n$)

$\Rightarrow Av = 7v \Rightarrow Av - 7v = 0 \Rightarrow (A - 7I)v = 0$

Therefore showing $Av = 7v$ for some $v \neq 0$ is equivalent to showing $(A-7I)v = 0$ has a nontrivial solution.

$$A - 7I = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}$$

So need to show $\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has nontrivial solution.

$$\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} t \\ t \end{pmatrix} \text{ for } t \in \mathbb{R}$$

So the system $(A-7I)v = 0$ has solution set $\text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ (what else is this called?)

$\Rightarrow$ 7 is an eigenvalue and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector (for example).

Note: the set $\text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ is called an eigenspace for 7

any $v \in \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ satisfies $Av = 7v$

this is a line here. (makes sense geometrically)

On your own: Let $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$. A has an eigenvalue 2. Find a basis for its eigenspace.

That is, find a basis for all v s.t. $Av = 2v$

Solution: This is equivalent to finding v s.t. $(A-2I)v = 0$.

$$A - 2I = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix}$$

$$(A-2I)v = 0 \Rightarrow \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$2v_1 = s - 6t \Rightarrow v_1 = \frac{1}{2}s - 3t$

$$\Rightarrow v = \begin{pmatrix} s - 3t \\ 2s \\ t \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} s + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} t \qquad s, t \in \mathbb{R}$$

$\Rightarrow$ a basis for the e-space of 2 is $\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$

Ex: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Find all eigenvalues and corresponding bases for eigenspaces

That is, $Av = \lambda v$

find first — λ will determine

Solution: $Av = \lambda v$ is equivalent to $(A - \lambda I)v = 0$

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$(A - \lambda I)v = 0 \quad \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\lambda \\ 0 & 1-\lambda^2 \end{pmatrix}$$

$\Rightarrow 1 - \lambda^2 = 0$

$\lambda = \pm 1$

Our possible eigenvalues are $\lambda = \pm 1$. We need to check that each eigenspace is non trivial.

$\lambda_1 = 1$: Find $v \in \mathbb{R}^2$ s.t. $Av = 1v$ or just $Av = v$