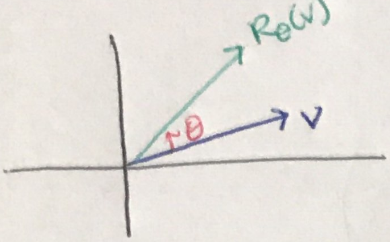
Ex (for all those who want visual examples!)

Let $V = W = \mathbb{R}^2$ and $T = R_\theta : V \to V$ defined by $R_\theta(v)$ = the vector obtained by rotating the vector v through an angle θ while preserving its length.

eg.



Exer for you: If $w = R_\theta(v)$ then $w = \|v\|(\cos(\varphi+\theta), \sin(\varphi+\theta))$ where φ is the angle v makes with the x axis. Check R_θ is a linear map.

R_θ is injective (if $w = R_\theta(u) = R_\theta(v)$ then rotating w by angle $-\theta$ gives $u = v$) and surjective (given w, $R_\theta(R_{-\theta}(w)) = w$.)

Then it has an inverse, $R_{-\theta}$ → verify that $R_\theta R_{-\theta} = R_{-\theta} R_\theta = I$.

How does this relate to matrices?

Def: An $n \times n$ matrix A is called invertible if there exists an $n \times n$ matrix B so that $AB = BA = I_n$ ← the $n \times n$ identity matrix $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & 1 \end{pmatrix}$

Thm: If A is invertible, the inverse of A is unique.

Proof: On your own.

Thm: If A is invertible, then $(A^{-1})^{-1} = A$.

Proof: Let $B = (A^{-1})^{-1}$. We want to show $B = A$.

By definition, $A^{-1}B = BA^{-1} = I$ and we know $A^{-1}A = AA^{-1} = I$.

So $A^{-1}B = I = A^{-1}A \Rightarrow A(A^{-1}B) = A(A^{-1}A)$

$(AA^{-1})B = (AA^{-1})A \Rightarrow IB = IA \Rightarrow B = A$.

(other ways to prove as well.)

Thm: If A, B are $n \times n$ invertible matrices then AB is also $n \times n$ and invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $AB(\ \ ) = I$

$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$ ☒

↑ Sometimes thought of as "socks-shoes" property

In practice, how do we know if a matrix is invertible?

Ex: $A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ if A^{-1} exists, call it $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

Then by definition, $AB = I_n \Rightarrow$ $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

That is, $A\begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

3 systems of equations, we usually find augmented matrix & row reduce to get solutions. $\left(A \,\middle|\, \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\left(A \,\middle|\, \begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$ & $\left(A \,\middle|\, \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$

We can solve all 3 systems at once:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \text{continue} \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right)$$

these are unique solutions to all 3 systems i.e. the entries of B.

Then $A^{-1} = B = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$

check: $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

$\therefore$ So: To find inverse, reduce $(A \mid I_n)$ to $(I_n \mid A^{-1})$

Thm: An $n \times n$ matrix is invertible $\Longleftrightarrow$ its row reduced echelon form is I_n.

Practice: Let's make a huge line of "if and only if" equivalencies.

Thm: $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$

if and only if

$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ has n pivots

if and only if

$\text{rref}\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = I_n$

if and only if

$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is invertible

if and only if

The linear transformation $T: V \to W$ associated with $\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is invertible

if and only if

T is an isomorphism.

if and only if

$\dim(V) = n = \dim(W)$

if and only if

T is injective and surjective

One More Ex: We said $\mathcal{L}(V,W) = \{$all linear maps from V to $W\}$ is a vector space.
To what space is it isomorphic? where $\dim(V) = n$, $\dim(W) = m$

Note: a linear transformation $T \in \mathcal{L}(V,V)$ is called a linear operator (since it only operates on V). If V is finite dimensional, what can you say about injectivity/surjectivity/invertibility of T? What if V is infinite dimensional?