MA 405-002: Introduction to Linear Algebra and Matrices, NCSU, Spring 2018

Written Homework #4

Due: Monday, April 2

Please indicate if you would like the grader to give feedback regarding your mathematical writing. This feedback will not be factored into your grade and only serves to help you grow as a mathematician.

1. Determine a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ that is not surjective. Is this the only such map? Make sure to justify your response.

2. Let B be a fixed $n \times n$ matrix. Define $T : M_{n,n} \to M_{n,n}$ by $T(A) = AB - BA$.
   (a) Prove T is a linear map.
   (b) Let $n = 2$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find $\mathcal{M}(T)$ and $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$.
   (c) Find a basis for $\mathrm{Im}(T)$ and $\ker(T)$ in the case of part (b).
   (d) For any n and any B, will T ever be injective? surjective?

3. Suppose that $S : \mathbb{R}^3 \to \mathbb{R}$ is defined by
$$S(x_1, x_2, x_3) = 2x_1 - x_2 + 3x_3.$$
Suppose also that $T : \mathbb{R} \to \mathbb{R}^3$ is given by
$$T(y) = (2y, -y, y).$$
What are the matrices of S, T, and $T \circ S$ with respect to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}$?

4. Let S and $T : U \to V$ be linear transformations.
   (a) If S and T are injective (surjective), is $S + T$ necessarily injective (surjective)? (Note that there are two questions.)
   (b) If S is injective (surjective) and $a \neq 0$, is aS necessarily injective (surjective)? (Note that there are two questions.)

5. Determine if the following linear transformations are injective, surjective, both, or neither. Make sure to justify your response by including the kernel and image corresponding to each map.
Note: You can use this to review by verifying that each of these are in fact linear maps!
   (a) $T : P_2 \to P_2$ defined by $T(p) = p' - p$.
   (b) $T : P_2 \to P_3$ defined by $T(p) = xp$.
   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = (x_1 + x_2, x_1 + x_2 + x_3, x_2 + x_3)$.
   (d) $T : P_2 \to \mathbb{R}^2$ defined by $T(p(x)) = (p(0), p'(0))$.