Solutions to Written Homework 3

① $B_1 = \{1, x, x^2\}$ $\quad B_3 = \{1+x-x^2,\ 4+x,\ 2-x+x^2\}$

a) From $B_1 \to B_3$ means write B_1 in terms of B_3:

$$1 = a_1(1+x-x^2) + a_2(4+x) + a_3(2-x+x^2)$$

$$1 = a_1 + 4a_2 + 2a_3 + (a_1 + a_2 - a_3)x + (-a_1 + a_3)x^2$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{Mathematica}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

The coefficient matrix will be the same for each vector in B_1.

$$x: \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 1 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{G-J}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4/3 \end{array}\right)$$

$$x^2: \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{\text{G-J}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Then COB matrix is

$$\left(\operatorname{rep}_{B_2}(1) \,\middle|\, \operatorname{rep}_{B_2}(x) \,\middle|\, \operatorname{rep}_{B_2}(x^2)\right) = \boxed{\begin{pmatrix} \frac{1}{3} & -\frac{4}{3} & -2 \\ 0 & 1 & 1 \\ \frac{1}{3} & -\frac{4}{3} & -1 \end{pmatrix}}$$

b) Then $\operatorname{rep}_{B_3}(1-2x+3x^2) = \begin{pmatrix} \frac{1}{3} & -\frac{4}{3} & -2 \\ 0 & 1 & 1 \\ \frac{1}{3} & -\frac{4}{3} & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}_{B_3}$ ✓

② a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow{\text{G-J}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ Then basis for column space is $\left\{\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}\right\}$ and the null space is $\{0\}$.

$\underset{\text{rank}}{2} + \underset{\text{nullity}}{0} = 2$ # columns

b) $\begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 2 & 0 \end{bmatrix} \xrightarrow{\text{G-J}} \begin{bmatrix} 1 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$\begin{array}{cccccc} \uparrow & \uparrow & & \uparrow & & \\ \multicolumn{6}{c}{\text{pivots}} \end{array}$$

Basis for column space is $\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right\}$ $= \mathbb{R}^3$ here.

Null space: $x_1 = -x_3 - \frac{1}{2}x_5 - \frac{1}{2}x_6$

$x_2 = -3x_3 - x_5 - x_6$

$x_4 = -\frac{1}{2}x_5 + \frac{1}{2}x_6$

$$= \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \,\middle|\, \downarrow \text{ hold}\right\} = \operatorname{span}\left\{\underset{x_3 = 1}{\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{x_5 = 2}{\begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}}, \underset{x_6 = 2}{\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}}\right\}$$

$\leftarrow$ Basis for null space

$\underset{\text{rank}}{3} + \underset{\text{nullity}}{3} = \underset{\text{\# columns}}{6}$ ✓

c) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 3 \\ 1 & 2 & 5 & 6 \end{bmatrix} \xrightarrow{\text{G-J}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{array}{cccc} \uparrow & \uparrow & & \end{array}$$

Basis for column space is $\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}\right\}$ rank $= 2$

Null space $= \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \,\middle|\, \begin{array}{l} x_1 = -x_3 \\ x_2 = -2x_3 - 3x_4 \end{array}\right\}$

$\downarrow$

Basis $= \left\{\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}\right\}$ nullity $= 2$

$2 + 2 = 4$ ✓

③ Find a basis for span$\{x^3+x, 2x^3+3x, 3x^3-x-1, x+2, x^3+x^2, x^2-8\}$.

Write coordinates in terms of $\mathcal{B} = \{x^3, x^2, x, 1\}$

$$\begin{array}{cccccc} x^3+x & 2x^3+3x & 3x^3-x-1 & x+2 & x^3+x^2 & x^2-8 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -8 \end{pmatrix} \xrightarrow{G-J} \begin{pmatrix} 1 & 0 & 0 & 20 & 0 & -91 \\ 0 & 1 & 0 & -7 & 0 & 33 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Then basis is $\{x^3+x, 2x^3+3x, 3x^3-x-1, x^3+x^2\}$

④ Find a basis of $\mathbb{R}^5$ containing $S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$S' = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{G-J} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}$$

Basis is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$