Prop: If V is an n dimensional vector space over $\mathbb{R}$, then $V \cong \mathbb{R}^n$.
Can you think of the isomorphism? (or one of).

Proof (sketch): Choose a basis for V, $\{v_1, v_2, \ldots, v_n\}$

Define $T(v) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ where $v = a_1v_1 + \cdots + a_nv_n$

Prove to yourself that this map is linear.
Is this bijective? What is $M(T)$?

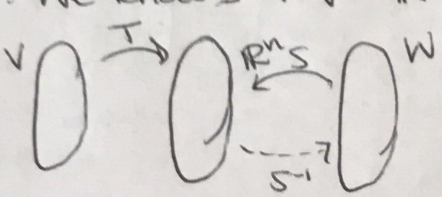
$$M(T) = \left(T(v_1) \,\middle|\, T(v_2) \,\middle|\, \cdots \,\middle|\, T(v_n) \right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

So $\dim(\ker(T)) = 0$ and $\dim(\operatorname{Im}(T)) = n = \dim(\mathbb{R}^n)$

$\therefore T$ is bijective and so $V \cong \mathbb{R}^n$ ☒ ← Everything we did with coordinates is okay!

Prop: If V and W are both n dimensional vector spaces over $\mathbb{R}$, then $V \cong W$

Proof (sketch): We know $\exists\ T: V \to \mathbb{R}^n$ and $S: W \to \mathbb{R}^n$



So $S^{-1} \circ T: V \to W$ is an isomorphism

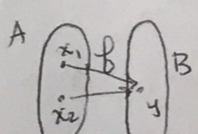
ex: $V = P_2$, $W = \{2\times 2 \text{ symmetric matrices}\}$.
Both are three dimensional vector spaces, so should be isomorphic as both are isomorphic to $\mathbb{R}^3$

Space:	P_2	$\mathbb{R}^3$	W
Elements:	$a + bx + cx^2 \longleftrightarrow$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longleftrightarrow$	$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

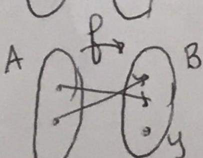
Def: A function $f: A \to B$ is <u>invertible</u> if $\exists$ function g so that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$

Prop: $f: A \to B$ is invertible iff f is injective and surjective.

Proof Sketch: There is a pretty detailed proof in Axler section 3D.



f not injective means the inverse won't be a function
$g(y) = x_1$ or x_2

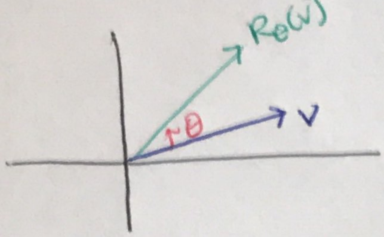


f not surjective means $g(y)$ not defined.

Ex (for all those who want visual examples!)

Let $V = W = \mathbb{R}^2$ and $T = R_\theta : V \to V$ defined by $R_\theta(v)$ = the vector obtained by rotating the vector v through an angle θ while preserving its length.

eg.



Exer for you: If $w = R_\theta(v)$ then $w = \|v\|(\cos(\varphi+\theta), \sin(\varphi+\theta))$ where φ is the angle v makes with the x axis. Check R_θ is a linear map.

R_θ is injective (if $w = R_\theta(u) = R_\theta(v)$ then rotating w by angle $-\theta$ gives $u = v$) and surjective (given w, $R_\theta(R_{-\theta}(w)) = w$.)

Then it has an inverse, $R_{-\theta}$ → verify that $R_\theta R_{-\theta} = R_{-\theta} R_\theta = I$.

How does this relate to matrices?

Def: An $n \times n$ matrix A is called invertible if there exists an $n \times n$ matrix B so that $AB = BA = I_n$ ← the $n \times n$ identity matrix $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & 1 \end{pmatrix}$

Thm: If A is invertible, the inverse of A is unique.

Proof: On your own.

Thm: If A is invertible, then $(A^{-1})^{-1} = A$.

Proof: Let $B = (A^{-1})^{-1}$. We want to show $B = A$.

By definition, $A^{-1}B = BA^{-1} = I$ and we know $A^{-1}A = AA^{-1} = I$.

So $A^{-1}B = I = A^{-1}A \Rightarrow A(A^{-1}B) = A(A^{-1}A)$

$(AA^{-1})B = (AA^{-1})A \Rightarrow IB = IA \Rightarrow B = A$.

(other ways to prove as well.)

Thm: If A, B are $n \times n$ invertible matrices then AB is also $n \times n$ and invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $AB(\quad) = I$

$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$ ☒

↑ Sometimes thought of as "socks-shoes" property