($\Leftarrow$) Suppose $\ker(T) = \{0\}$. We want to show that if $T(u) = T(v)$, then we must have $u = v$.

Suppose $T(u) = T(v)$. Then $T(u) - T(v) = 0$

So $T(u - v) = 0$ by linearity of T

$\Rightarrow u - v \in \ker(T)$.

By assumption, $\ker(T) = \{0\} \Rightarrow u - v = 0$ and so $u = v$.

Therefore, T is injective if and only if $\ker(T) = \{0\}$. $\blacksquare$

On your own: $T: M_{2\times2} \to M_{2\times2}$

$$T\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix} \quad T\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad T\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1&1\\0&0\end{pmatrix} \quad T\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\2&0\end{pmatrix}$$

① Find $T\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $M(T)$.

② Is T injective and/or surjective?

Solution: ① $T\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}b+c&a+c\\2d&-b\end{pmatrix}$ $\quad M(T) = \begin{pmatrix}0&1&1&0\\1&0&1&0\\0&0&0&2\\0&-1&0&0\end{pmatrix}$

② $$M(T) = \begin{pmatrix}0&1&1&0\\1&0&1&0\\0&0&0&2\\0&-1&0&0\end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix}1&0&1&0\\0&1&1&0\\0&0&0&2\\0&-1&0&0\end{pmatrix} \xrightarrow[R_3]{R_4 \leftrightarrow R_2} \begin{pmatrix}1&0&1&0\\0&-1&0&0\\0&1&1&0\\0&0&0&2\end{pmatrix} \xrightarrow[R_2+R_3]{-R_2} \begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&2\end{pmatrix}$$

upper triangular will have 4 pivots

$\dim(\mathrm{col}(M(T))) = \dim(\mathrm{Im}(T)) = 4 \Rightarrow$ T is surjective

$= \dim(M_{2\times2})$

rank + nullity = 4

$4 + 0 = 4$

nullity $= \dim(\ker(T)) = 0 \Rightarrow T$ is injective.

Ex: Same questions for $T: P_2 \to P_3$ where $T(f) = x^2 f'' - 2f' + xf$

Solution: ① Let $f = a + bx + cx^2$. Then $T(a+bx+cx^2) = 2cx^2 - 2(b+2cx) + x(a+bx+cx^2)$

$f' = b + 2cx$ $\qquad = -2b + (a-4c)x + (b+2c)x^2 + cx^3$

$f'' = 2c$

So $M(T)\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-2b\\a-4c\\b+2c\\c\end{pmatrix} \Rightarrow M(T) = \begin{pmatrix}0&-2&0\\1&0&-4\\0&1&2\\0&0&1\end{pmatrix}$

$\mathcal{B} = \{1, x, x^2\}$

rank$(M(T)) = 3$ so not surjective since $\dim(P_3) = 4$.

no free variables $\Rightarrow$ nullity $= 0$

So T is injective.

WORKSHEET

Def: Let V and W be vector spaces. If there is a linear map $T: V \to W$ that is bijective, we say V and W are isomorphic and T is an isomorphism

Notation: $V \cong W$

Note: Injective + Surjective = bijective (sets)

Injective + Surjective + Linear = isomorphism (vector spaces)