Ex: $h: \mathbb{N} \to \mathbb{N} = \{1, 2, 3, \ldots\}$

$x \mapsto x+1$

Injective: if $h(x) = h(y) \Rightarrow x+1 = y+1 \Rightarrow x = y$

Not surjective: $1 \notin \text{range}(h)$.

Can we "fix" these functions to make them onto?

First Ex: $F: \mathbb{R} \to [0, \infty)$ $\quad F(x) = x^2$. is surjective!

we can go further... $F: [0, \infty) \to [0, \infty)$ is a bijection.

Back to 4os... $\quad T: \mathbb{R}^3 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 2y + 3z$

Injective? No. For example, $T\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2 = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Surjective? yes, we showed $\text{range}(T) = \mathbb{R}$

- Int: $P_2 \to P_3$ defined by $p \mapsto \int_0^x p(t)\,dt$

Injective? Assume $\int_0^x p(t)\,dt = \int_0^x q(t)\,dt$. Differentiating both sides, by the FTC (part 1),

we obtain: $p(x) = q(x)$ yes!

Surjective? No. For example, we cannot obtain $x + 1 \in P_3$

- $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Injective? What if $T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$?

$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = T\left(u_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = u_1 T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{by linearity of } T$$

$$= u_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 + u_2 \\ -u_2 \end{pmatrix}$$

OR, faster: find $M(T) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

Then $T(v) = M(T)(v) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ -v_2 \end{pmatrix}$

Either way, $\begin{pmatrix} u_1 + u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ -v_2 \end{pmatrix}$ So $\left.\begin{array}{l} u_1 + u_2 = v_1 + v_2 \\ -u_2 = -v_2 \end{array}\right\} \Rightarrow \begin{array}{l} u_1 = v_1 \\ u_2 = v_2 \end{array}$ $\therefore$ T is injective

Surjective? $T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ what values of w_1, w_2 are possible?

$\begin{pmatrix} u_1 + u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Rightarrow T\begin{pmatrix} w_1 + w_2 \\ -w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ yes surjective

OR Easier: $\text{Rank}(M(T)) = 2 \Rightarrow \text{Im}(T)$ is a 2-dimensional subspace of $\mathbb{R}^2$

$\Rightarrow \text{Im}(T) = \mathbb{R}^2$ So $\therefore$ T is surjective

Takeaway: $T: V \to W$ is surjective if $\dim(W) = \dim(\text{range}(T))$

$= \dim(\text{col}(M(T))) = \#\text{ pivots} = \text{rank}(M(T))$

Prop: If $T: V \to W$ is a linear transformation, T is injective iff $\ker(T) = \{0\}$.

Proof: ($\rightarrow$) Suppose T is injective. Then whenever $T(u) = T(v)$, we have $u = v$.

Let $v \in \ker(T)$. Then $T(v) = 0$. We've shown $T(0) = 0$ for all linear maps

Thus, $v = 0$.

($\Leftarrow$) Suppose $\ker(T) = \{0\}$. We want to show that if $T(u) = T(v)$, then we must have $u = v$.

Suppose $T(u) = T(v)$. Then $T(u) - T(v) = 0$

So $T(u-v) = 0$ by linearity of T

$\Rightarrow u - v \in \ker(T)$.

By assumption, $\ker(T) = \{0\} \Rightarrow u - v = 0$ and so $u = v$.

Therefore, T is injective if and only if $\ker(T) = \{0\}$. $\boxtimes$