Notice anything?

$\ker(T) = \text{null}(\mathcal{M}(T))$ and $\text{range}(T) = \text{col}(\mathcal{M}(T))$!

Prop: Let $T: V \to W$ be a linear transformation. Then
① $\ker(T)$ is a subspace of V
② $\text{Range}(T)$ is a subspace of W

Proof: ① We check using the subspace proposition.
$0 \in \ker(T)$ since we proved earlier that $T(0) = 0$.
We check that $\ker(T)$ is closed under addition. Let $v_1, v_2 \in \ker(T)$. Then
$T(v_1) = T(v_2) = 0$. $T(v_1 + v_2) = T(v_1) + T(v_2)$ by the additivity of T
$= 0 + 0 = 0$. So $v_1 + v_2 \in \ker(T)$
Then $\ker(T)$ is closed under addition.
Finally, we need to check $\ker(T)$ is closed under scalar multiplication.
Let $\lambda \in F$, $v \in \ker(T)$. $T(\lambda v) = \lambda T(v)$ by homogeneity of T
$= \lambda \cdot 0$ since $v \in \ker(T)$
$= 0$ since W is a vector space
Then $\ker(T)$ is closed under scalar multiplication.
Therefore, $\ker(T)$ is a subspace of V.
② on your own → Review for final! ☒

Recall: For M any $n \times m$ matrix

$$M = n\left\{\underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}_{m}\right.$$

$(\#\text{ pivots}) + (\#\text{ free variables}) = \underbrace{(\#\text{ columns of } M)}_{m}$ in RREF of M

is equivalent to

$\text{rank}(M) + \text{nullity}(M) = m$

is equivalent to...

Fundamental Theorem of Linear Maps: Suppose V is finite dimensional and $T \in \mathcal{L}(V, W)$. Then $\text{range}(T)$ is finite-dimensional and

$$\dim(\text{range}(T)) + \dim(\ker(T)) = \dim V$$

(equivalent in the case $\dim W < \infty$ also).

Why equivalent in this case? Think of $\mathcal{M}(T)$.

Ex: Define $T: \mathbb{R}^3 \to \mathbb{R}$ by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x + 2y + 3z$

① Find $\ker(T)$.
By definition, $\ker(T) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + 2y + 3z = 0 \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x = -2y - 3z \right\} = \left\{ \begin{pmatrix} -2s - 3t \\ s \\ t \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

OR: make $\mathcal{M}(T)$ & find null space.

$T\begin{pmatrix}1\\0\\0\end{pmatrix}=1 \quad T\begin{pmatrix}0\\1\\0\end{pmatrix}=2 \quad T\begin{pmatrix}0\\0\\1\end{pmatrix}=3$ so $\mathcal{M}(T) = \begin{pmatrix}1 & 2 & 3\end{pmatrix} \Rightarrow \text{Null}(\mathcal{M}(T)) = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \middle| x=-2y-3z\right\}$

② Find range(T).

By definition, range$(T) = \{T(v) \mid v \in \mathbb{R}^3\}$

$\Rightarrow T\begin{pmatrix}x\\y\\z\end{pmatrix} = k$, what values of k are possible? (this kind of feels like span)

we can do $T\begin{pmatrix}k\\0\\0\end{pmatrix} = k$, so k can be anything (all of $\mathbb{R}$)

OR by $\mathcal{M}(T)$: range$(T) = \text{col}(\mathcal{M}(T)) = \text{span}\{1,2,3\} = \text{span}\{1\}$.

Fundamental Thm: $\dim(\text{range}) + \dim(\ker) = m$

$$1 + 2 = 3 \checkmark$$

Ex Let $V = P_3$. Define $D: V \to V$ by $D(p(x)) = p'(x)$. Determine $\ker(D)$ and range(D).

Solution: By definition, $\ker(D) = \{v \in V \mid D(v) = 0\}$ "the set of polynomials in P_3 whose derivative is 0"

From calculus, we know $\ker(D) = \text{span}\{1\} = \{\text{constant polynomials}\}$

We also know we can obtain any polynomial in P_2 from differentiating the appropriate polynomial in P_3: $D(ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3) = a + bx + cx^2$

Looking at $\mathcal{M}(D) = \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}$. It's easy to see that RREF is

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$\downarrow$ free variable $\quad \uparrow\uparrow\uparrow$ pivots

So range$(D) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{B}_2}, \begin{pmatrix}0\\2\\0\end{pmatrix}_{\mathcal{B}_2}, \begin{pmatrix}0\\0\\3\end{pmatrix}_{\mathcal{B}_2}\right\}$

$= \text{span}\{1, 2x, 3x^2\} = P_2$.

and $\ker(D)$ is determined by free variables $\Rightarrow$ the first basis vector of P_3 can be anything $\Rightarrow \ker(D) = \text{span}\{1\} = P_0$.

On your Own: What if we changed V? or made $D: V \to W$? What are the viable options? Do they change $\ker(D)$? range(D)? What if $D: P(\mathbb{R}) \to P(\mathbb{R})$ where $P(\mathbb{R}) = \{\text{all polynomials over } \mathbb{R}\}$?

$\uparrow$ infinite vector space

Ex $T: \mathbb{R}^3 \to \mathbb{R}^3$ linear map given some basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$, $T(v_1) = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $T(v_2) = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, $T(v_3) = \begin{pmatrix}2\\1\\-1\end{pmatrix}$

① Is $\begin{pmatrix}1\\2\\1\end{pmatrix} \in$ range(T)?

② Find bases for range(T) and $\ker(T)$.

Solution: ①. By definition, this is the same as asking

"Is there a v such that $T(v) = \begin{pmatrix}1\\2\\1\end{pmatrix}$?"

If so, $v = av_1 + bv_2 + cv_3$ since $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

$\Rightarrow T(v) = T(av_1 + bv_2 + cv_3) = aT(v_1) + bT(v_2) + cT(v_3)$ by linearity of T.