More on Linear Maps

Recall: $T: V \to W$ linear map means $\cdot\, T(v_1+v_2) = Tv_1 + Tv_2 \;\forall\, v_1, v_2 \in V$

$\cdot\, T(\lambda v) = \lambda Tv \;\forall\, \lambda \in F, v \in V$

Def: Suppose $S, T \in \mathcal{L}(V,W)$ and $\lambda \in F$. The sum $S+T$ and the product λT are also linear maps from V to W defined by $(S+T)v = Sv + Tv$ and $(\lambda T)v = \lambda(Tv)$ for all $v \in V$. With these, $\mathcal{L}(V,W)$ is a vector space (Review: verify!)

(Review: verify!)

what's $0 \in \mathcal{L}(V,W)$?

We can do more with $\mathcal{L}(V,W)$ than with other vector spaces:

Def: If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ then the product $ST \in \mathcal{L}(U,W)$ is $(ST)(u) = S(Tu)$ for all $u \in U$.

$$U \xrightarrow{T} V \xrightarrow{S} W$$

verify!

Composition of functions

★ NOTE: the target of T must be part of the domain of S.

Properties of Products of Linear Maps

- $(T_1T_2)T_3 = T_1(T_2T_3)$ whenever $T_3 \in \mathcal{L}(U,V)$, $T_2 \in \mathcal{L}(V,W)$, $T_1 \in \mathcal{L}(W,Y)$ (associativity)
- $\underset{\mathcal{L}(V,V)}{TI} = \underset{\mathcal{L}(W,W)}{IT} = T$ for $T \in \mathcal{L}(V,W)$ and I the identity map $(I(v) = v \;\forall\, v \in V)$
- $(S_1+S_2)T = S_1T + S_2T$ and $S(T_1+T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U,V)$ and $S, S_1, S_2 \in \mathcal{L}(V,W)$ (Distributive properties)

★ Note: multiplication of linear maps is not necessarily commutative

Non 405 Example: $f = x^2$ $g = \sin(x)$

Ex: We showed that both $D: P_3 \to P_2$ defined by $D(p(x)) = p'(x)$ and $\text{int}: P_2 \to P_3$ defined by $\text{int}(p(x)) = \int_0^x p(t)\,dt$ are linear maps. Is it necessarily true that $D\,\text{int} = \text{int}\,D$?

Solution: No. Consider the polynomial $5 - 3x + 2x^2$.

$(D\,\text{int})(5-3x+2x^2) = D(\text{int}(5-3x+2x^2)) = D(5x - \frac{3}{2}x^2 + \frac{2}{3}x^3) = 5 - 3x + 2x^2$

$(\text{int}\,D)(5-3x+2x^2) = \text{int}(D(5-3x+2x^2)) = \text{int}(-3+4x) = -3x + 2x^2$ $\neq$

Relating all this back to matrices...

If the same bases are used to determine both $\mathcal{M}(T)$ and $\mathcal{M}(S)$ and $\mathcal{M}(S+T)$,

we have: $\cdot\, \mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T) \;\forall\, S, T \in \mathcal{L}(V,W)$

$\cdot\, \mathcal{M}(\lambda T) = \lambda \mathcal{M}(T) \;\forall\, \lambda \in F, T \in \mathcal{L}(V,W)$

Axler's notation for matrix of a linear map

and for using the same bases in appropriate spaces, we have:

$\cdot\, \mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$ for $T \in \mathcal{L}(U,V)$ $S \in \mathcal{L}(V,W)$

Back to our example: Using the basis $\{1, x, \ldots, x^n\}$ for P_n, we have:

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \text{ and } \mathcal{M}(\text{Int}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Find $\mathcal{M}(D\,\text{Int})$ and $\mathcal{M}(\text{Int}\,D)$

DInt: check $1, x, x^2$:

$$M(\text{DInt}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Is this $M(D)M(\text{Int})$?

IntD: check $1, x, x^2, x^3$

$$M(\text{IntD}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Is this $M(\text{Int})M(D)$?

Then multiplication of linear maps is multiplication of their matrices. Neither commute!

Remember, To find $T(v)$, we cand do $M(T) \cdot [v]_{\mathcal{B}}$.

Prop: Let $T \in \mathcal{L}(V, W)$. Then $T(0) = 0$.

Proof (Different from Axler): Consider $T(0)$. This equals $T(v - v)$ $\forall v \in V$.

Then $T(v-v) = T(v + (-1)v) = T(v) + T((-1)v)$ by additivity of T

$= T(v) + (-1)T(v)$ by homogeneity of T

$= 0$ by additive inverses in W.

Notice:

$$\text{DInt}(a + bx + cx^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bx + cx^2$$

and

$$\text{IntD}(a + bx + cx^2 + dx^3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \\ d \end{pmatrix} = bx + cx^2 + dx^3$$

Check this against $D(\text{Int}(a + bx + cx^2))$ and $\text{Int}(D(a + bx + cx^2 + dx^3))$

Def: For $T \in \mathcal{L}(V, W)$, the null space of T (or the kernel of T) is $\{v \in V \mid T(v) = 0\}$

The range of T is $\{T(v) \mid v \in V\} = \{w \in W \mid \exists v \in V \text{ with } T(v) = w\}$

oyo

Review: ① Find $M(T)$ where T is the linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4, x_5) = (x_1 - 2x_2 + 3x_4,\ x_3 - 5x_4,\ 0)$

wrt standard bases if not specified

② Find $\ker(T)$ & $\text{range}(T)$.

Solution: ①

$$M(T) = \begin{pmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

② $\ker(T) = \{v \in \mathbb{R}^5 \mid T(v) = 0\}$

$T(x_1, x_2, x_3, x_4, x_5) = (x_1 - 2x_2 + 3x_4,\ x_3 - 5x_4,\ 0) = (0, 0, 0)$

$\Rightarrow x_1 - 2x_2 + 3x_4 = 0 \quad \to x_1 = 2x_2 - 3x_4$

$x_3 - 5x_4 = 0 \quad x_3 = 5x_4$

$0 = 0 \quad x_5$ anything

$$\therefore \ker(T) = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$\text{range}(T) = \{(\underbrace{x_1 - 2x_2 + 3x_4}_{\text{anything}},\ \underbrace{x_3 - 5x_4}_{\text{anything}},\ 0)\}$$

(think $x_2 = x_4 = 0$, x_1, x_3 anything)

$$= \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$