## Definitions

(I) A matrix is said to be in *row echelon form* **(REF)** if:

(a) The left-most nonzero entry in each row is 1 (which will be called "the leading 1").

(b) The entries below any leading 1 are all 0.

(c) The leading 1 for each row is to the left of the leading 1 for any row below it.

(d) Any row whose entries are all 0 is located below the rows that have leading 1's.

The matrix $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is in row echelon form, while the matrix $\begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & -2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is not in row echelon form (condition (c) is not satisfied).

(II) A matrix is said to be in *reduced row echelon form* **(RREF)** if, in addition to having the properties of REF, it also has the property:

(e) The entries above any leading 1 are all 0.

The matrix $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is in reduced row echelon form, while the matrix $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is not in reduced row echelon form (condition (e) is not satisfied).

**Theorem**

Every matrix can be transformed into a REF matrix or a RREF matrix by a sequence of elementary row operations.

The REF of a matrix is not unique, but the RREF of a matrix is unique.

### Gauss Elimination with back substitution/back addition

Gauss Elimination (with back substitution or back addition) is a systematic procedure for solving systems of linear equations; elementary row operations are performed on the augmented matrix of the system to convert it into a much "simpler" matrix, either the row echelon form or reduced row echelon form. The solution of the system can then be easily determined by inspecting either of these matrices.

1. Form the augmented matrix of the system.

2. Use elementary row operations to convert the augmented matrix to REF:

   (a) Choose a non-zero entry (pivot) in the first non-zero column (starting from the left). If the chosen entry is not in the top row, a row interchange will be necessary.

(b) If the chosen pivot in the top row is not 1, replace the top row by the row obtained from dividing it by the pivot. The new top row will have the leftmost entry 1, called the leading 1.
(c) Apply elementary row operations to make all the entries below the leading 1 to be 0.
(d) Move the zero rows (if any) to the bottom of the matrix.
(e) Cover the top row and start again with (a).

Stop when the matrix is in row echelon form.

3. Write the system of linear equations corresponding to the REF matrix, then use **back substitution** to solve the system.

   **OR**

   Perform **back-addition** to find the RREF of the augmented matrix, that is, starting with the bottom non-zero row and moving upwards, use elementary row operations to make zeros above all leading 1's, then identify the solution of the initial system from the system of linear equations corresponding to the RREF matrix.

## Gauss-Jordan Elimination

Gauss-Jordan elimination is another systematic procedure for solving systems of linear equation by directly reducing the augmented matrix of the system to its RREF. The only distinction between these two methods is in part (c) and the final step of identifying the solutions.

1. Form the augmented matrix of the system.
2. Use elementary row operations to convert the augmented matrix to RREF:
   (a) Choose a non-zero entry (pivot) in the first non-zero column (starting from the left). If the chosen entry is not in the top row, a row interchange will be necessary.
   (b) If the chosen pivot in the top row is not 1, replace the top row by the row obtained from dividing it by the pivot. The new top row will have the leftmost entry 1, called the leading 1.
   (c) Apply elementary row operations to make all the entries above **and below** the leading 1 to be 0.
   (d) Move the zero rows (if any) to the bottom of the matrix.
   (e) Cover the top row and start again with (a).

   Stop when the matrix is in REDUCED ROW ECHELON FORM.
3. The solution of the initial system is now easy to identify from the system of linear equations corresponding to the RREF matrix.