Solutions to Written HW 2

① a) $x_1 + 2x_2 - x_3 + x_4 = 0$
$-3x_1 + x_3 + 2x_4 = 0.$

Solution set is $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \begin{array}{c} x_1 + 2x_2 - x_3 + x_4 = 0 \\ \text{and} \\ -3x_1 + x_3 + 2x_4 = 0 \end{array} \right\}$

Solve using G-J: $\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ -3 & 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{3R_1+R_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 6 & -2 & 5 & 0 \end{array}\right) \xrightarrow{\frac{1}{6}R_2}$

$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -2/6 & 5/6 & 0 \end{array}\right) \xrightarrow{-2R_2+R_1} \left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{5}{6} & 0 \end{array}\right)$

$x_1 - \frac{1}{3}x_3 + -\frac{2}{3}x_4 = 0 \Rightarrow x_1 = \frac{1}{3}x_3 + \frac{2}{3}x_4$

$x_2 - \frac{1}{3}x_3 + \frac{5}{6}x_4 = 0 \Rightarrow x_2 = \frac{1}{3}x_3 - \frac{5}{6}x_4.$

Solutions look like

$x_3 = s$
$x_4 = t$

$\begin{pmatrix} \frac{1}{3}s + \frac{2}{3}t \\ \frac{1}{3}s - \frac{5}{6}t \\ s \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} \frac{2}{3} \\ -\frac{5}{6} \\ 0 \\ 1 \end{pmatrix} t$

So a spanning set is $\left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{5}{6} \\ 0 \\ 1 \end{pmatrix} \right\}$

b) yes! It's a linearly independent spanning set.

c) 2 (number of elements in basis)

d) 2 → x_3 and x_4 do not have pivots

② Let $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = p(x) \in P_4$.

Then $p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$

$p''(x) = 2a_2 + 6a_3 x + 12a_4 x^2$

$p(1) = a_0 + a_1 + a_2 + a_3 + a_4 = 0$
$p'(0) = a_1 = 0$
$p''(0) = 2a_2 = 0$

$\rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$ row echelon form!

$a_0 = -a_3 - a_4$

$p(x) \in W$ if $p(x) = (-a_3 - a_4) + a_3 x^3 + a_4 x^4$ (this is the span)

Easy way to find a basis → set $a_3 = 1,\ a_4 = 0$ $\quad p_1(x) = -1 + x^3$

set $a_3 = 0,\ a_4 = 1$ $\quad p_2(x) = -1 + x^4$

$\{x^3 - 1, x^4 - 1\}$

③ a) Augmented:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 2 & 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 0 & 14 \end{array}\right) \rightsquigarrow$$

RREF:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad 0 \neq 1$$

pivot in RHS

$\boxed{\text{No Solution}}$

b) Augmented:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & -1 & 0 & 0 \end{array}\right) \rightsquigarrow$$

RREF:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

pivot in each column in coefficient

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

c) Augmented:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 2 & 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 0 & 6 \end{array}\right) \longrightarrow$$

RREF:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

no pivots, free variables.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \left\{ \begin{pmatrix} 4-s-t \\ -6+2s+3t \\ s \\ t \end{pmatrix}, \ s,t \in \mathbb{R} \right\}$$