the change of basis matrix & using matrix multiplication

To move <u>from</u> $\mathcal{B}_1 = \{u_1, \dots, u_n\}$ to $\mathcal{B}_2$, we use $\left(\begin{array}{c|c|c|c} | & | & & | \\ rep_{\mathcal{B}_2}(u_1) & rep_{\mathcal{B}_2}(u_2) & \cdots & rep_{\mathcal{B}_2}(u_n) \\ | & | & & | \end{array} \right)$

Example Problems:

- Find $rep_{\mathcal{B}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ where $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$
- Find the COB matrix from $\mathcal{B}$ to the standard basis for $\mathbb{R}^3$ (and vice versa).
- Multiply the two matrices from ↑ together. What do you get?

Linear maps: functions $T: V \to W$ with $T(u+v) = Tu + Tv$ $\forall u, v \in V$ and $T(\lambda v) = \lambda Tv$ $\forall v \in V, \lambda \in F$.

Example Problems:

- Show the map $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ -x_2 \end{pmatrix}$ is linear.
- Find $[T]_{\mathcal{B}_1, \mathcal{B}_2}$ where T is defined as above and $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$
- Find $T\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (the image of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ under T).

Solutions to Example Review Problems

- $\{p \in P_3 \mid p''(1) - p'(2) - p(3) = 0\}$. Arbitrary element?

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\quad p'(x) = a_1 + 2a_2 x + 3a_3 x^2$ $\quad p''(x) = 2a_2 + 6a_3 x$

$p(3) = a_0 + 3a_1 + 9a_2 + 27a_3$ $\quad p'(2) = a_1 + 4a_2 + 12a_3$ $\quad p''(1) = 2a_2 + 6a_3$

$p''(1) - p'(2) - p(3) = 2a_2 + 6a_3 - a_1 - 4a_2 - 12a_3 - a_0 - 3a_1 - 9a_2 - 27a_3 = 0$

$-33a_3 - 11a_2 - 4a_1 - a_0 = 0 \Rightarrow a_0 = -33a_3 - 11a_2 - 4a_1$

Basis is $\{-33 + x^3, -11 + x^2, -4 + x\}$

- S is not a basis for M_2. $\dim(M_2) = 4$ and $|S| = 5$, so S cannot be linearly independent
- $\left(\begin{array}{ccc|c} 0 & 1 & -3 & 4 \\ 1 & -2 & 1 & 0 \\ -4 & 1 & -1 & 7 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 4 \\ -4 & 1 & -1 & 7 \end{array} \right)$ OR $\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -4 & 1 & -1 & 7 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 4 \end{array} \right)$
- $x_1 + 4x_3 = -1$
  $x_2 - 2x_3 + x_5 = 5$
  $x_4 = 0$

  $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1-4s \\ 5+2s-t \\ s \\ 0 \\ t \end{pmatrix} \right\}$ $s, t \in \mathbb{R}$

  $\Rightarrow x_1 = -1 - 4x_3$
  $x_2 = 5 + 2x_3 - x_5$
  $x_4 = 0$

  $= \left\{ \begin{pmatrix} -1 \\ 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$
- $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 6 \end{pmatrix} \right\}$
- No Solutions.
- Basis for column space $= \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} \right\}$

  Null space $= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s + \frac{10}{7}t \\ -s + \frac{1}{7}t \\ s \\ \frac{2}{7}t \\ t \end{pmatrix} \right\}$ Basis $= \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 1 \\ 0 \\ 2 \\ 7 \end{pmatrix} \right\}$

- $\begin{pmatrix}1\\2\\3\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\quad \begin{matrix}a+b=1\\a+b+c=2\\b-c=3\end{matrix}$ $\quad$ RREF $= \left(\begin{array}{ccc|c}1&0&0&-3\\0&1&0&4\\0&0&1&1\end{array}\right) \Rightarrow rep_{\mathcal{B}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-3\\4\\1\end{pmatrix}_{\mathcal{B}}$

- From $\mathcal{B}$ to standard: $\qquad$ From standard to $\mathcal{B}$:

$$\begin{pmatrix}1&1&0\\1&1&1\\0&1&-1\end{pmatrix}_{\mathcal{B},S} \qquad \left(rep\begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{B}} \middle| rep_{\mathcal{B}}\begin{pmatrix}0\\1\\0\end{pmatrix} \middle| rep_{\mathcal{B}}\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}2&-1&-1\\-1&1&1\\-1&1&0\end{pmatrix}_{S,\mathcal{B}}$$

- $\begin{pmatrix}1&1&0\\1&1&1\\0&1&-1\end{pmatrix}\begin{pmatrix}2&-1&-1\\-1&1&1\\-1&1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ identity matrix.

- ① Let $x, y \in \mathbb{R}^2$: $T(x+y) = T\left(\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}y_1\\y_2\end{pmatrix}\right) = T\left(\begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix}\right)$ since $\mathbb{R}^2$ is v. space

Note, T is well-defined

$$= \begin{pmatrix}(x_1+y_1)+(x_2+y_2)\\(x_1+y_1)-(x_2+y_2)\\-(x_2+y_2)\end{pmatrix} \text{ by def of } T$$

$$= \begin{pmatrix}x_1+x_2\\x_1-x_2\\-x_2\end{pmatrix} + \begin{pmatrix}y_1+y_2\\y_1-y_2\\-y_2\end{pmatrix} \text{ since } \mathbb{R}^3 \text{ v. space}$$

$$= T(x) + T(y) \text{ by def of } T$$

② Let $x \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$. $T(\lambda x) = T\left(\lambda\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)$

$$= T\left(\begin{pmatrix}\lambda x_1\\\lambda x_2\end{pmatrix}\right) \text{ since } \mathbb{R}^2 \text{ v. space}$$

$$= \begin{pmatrix}\lambda x_1+\lambda x_2\\\lambda x_1-\lambda x_2\\-\lambda x_2\end{pmatrix} \text{ by def of } T$$

$$= \lambda\begin{pmatrix}x_1+x_2\\x_1-x_2\\-x_2\end{pmatrix} \text{ since } \mathbb{R}^3 \text{ v. space}$$

$$= \lambda T(x) \text{ def of } T.$$

- $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$ $\Rightarrow [T]_{\mathcal{B}_1,\mathcal{B}_2} = \begin{pmatrix}1&1\\1&-1\\0&-1\end{pmatrix}_{\mathcal{B}_1,\mathcal{B}_2}$

- $\begin{pmatrix}1&1\\1&-1\\0&-1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\1\\-1\end{pmatrix} = T\left(\begin{pmatrix}2\\1\end{pmatrix}\right)$ by def

$$= 2T\begin{pmatrix}1\\0\end{pmatrix} + T\begin{pmatrix}0\\1\end{pmatrix}$$