② Let $\lambda \in F$, $(x,y,z) \in \mathbb{R}^3$ ($\lambda \in F \to \mathbb{R}$)

$$T(\lambda(x,y,z)) = T((\lambda x, \lambda y, \lambda z)) = (\lambda x + 4(\lambda y) + 2(\lambda z), \lambda x + \lambda y - \lambda z, -\lambda x + \lambda z)$$
$$= \lambda(x+4y+2z, x+y-z, -x+z)$$
$$= \lambda T(x,y,z) \checkmark$$

Does this look familiar? Can we write it another way?

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{input into } T}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\substack{\uparrow \\ \text{output / range}}}{\begin{pmatrix} x + 4y + 2z \\ x + y - z \\ -x + z \end{pmatrix}}$$

In general, any linear map from F^n to F^m is of the form

$$T(x_1, \ldots, x_n) = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \ldots, a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)$$

$$\Rightarrow T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Prop: Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T: V \to W$ s.t. $Tv_j = w_j$ for all $j = 1, \ldots, n$

Proof in Axler, but the basic idea is that if we know or define where a map sends the basis, it determines where everything else is sent. (By additivity + homogeneity + $V = \mathrm{span}(v_1, \ldots, v_n)$).

The map: $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$

Ex Let $V = W = P_2$. $\mathcal{B} = \{1, 1+x, 1+x+x^2\}$. We define T by $T(1) = x$, $T(1+x) = x^2$ and $T(1+x+x^2) = 1$. If we insist T is linear, then this is a linear transformation. Since we defined T on a basis, we see where it sends any $p \in P_2$:

First, let $p(x) = a_0 + a_1x + a_2x^2$. Then in terms of $\mathcal{B}$, we can express p as

$$p(x) = (a_0 - a_1)1 + (a_1 - a_2)(1+x) + a_2(1+x+x^2).$$

Then $T(p(x)) = T((a_0-a_1)1 + (a_1-a_2)(1+x) + a_2(1+x+x^2))$.

$= (a_0 - a_1)T(1) + (a_1 - a_2)(T(1+x)) + a_2T(1+x+x^2)$ since T is linear

$= (a_0 - a_1)x + (a_1 - a_2)x^2 + a_2(1)$

Ex. ① Show that the map $\mathrm{Int}: P_2 \to P_3$ defined by $\mathrm{Int}(p) = \int_0^x p(t)\,dt$ is a linear map

antiderivative of p s.t. $p(0) = 0$.

② Let $\mathcal{B}_1 = \{1, x, x^2\}$ and $\mathcal{B}_2 = \{1, x, x^2, x^3\}$ be bases for P_2, P_3 respectively. Find the coordinates of the image of every vector in $\mathcal{B}_1$ with respect to $\mathcal{B}_2$.

③ What is $\mathrm{Int}(4 + 6x - 9x^2)$? How many ways can you find this?

Solution: ① First, note that the map is well-defined. Every polynomial in P_2 will map to only one polynomial in P_3 with constant term 0.

Let $p_1, p_2 \in P_2$. Then $\mathrm{Int}(p_1 + p_2) = \int_0^x (p_1+p_2)(t)\,dt = \int_0^x p_1(t) + p_2(t)\,dt = \int_0^x p_1(t)\,dt + \int_0^x p_2(t)\,dt$

$= \mathrm{Int}(p_1) + \mathrm{Int}(p_2) \checkmark$

Now, let $p \in P_2$, $\lambda \in \mathbb{R}$. Then
$$\begin{aligned} Int(\lambda p) &= \int_0^x (\lambda p)(t)\,dt \\ &= \int_0^x \lambda(p(t))\,dt \\ &= \lambda \int_0^x p(t)\,dt = \lambda\, Int(p) \checkmark \end{aligned}$$

Therefore, $Int(p)$ is a linear map.

② $Int(1) = \int_0^x 1\,dt = x = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}_{B_2}$. $\quad Int(x) = \int_0^x t\,dt = \frac{x^2}{2} = \begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}_{B_2}$

$Int(x^2) = \int_0^x t^2\,dt = \frac{x^3}{3} = \begin{pmatrix}0\\0\\0\\\frac{1}{3}\end{pmatrix}_{B_2}$.

③ one option: By definition of Int: $Int(4+6x-9x^2) = \int_0^x 4+6t-9t^2\,dt$
$$= 4x + 3x^2 - 3x^3$$

Another, by linearity of Int: $Int(4+6x-9x^2) = 4\,Int(1) + 6\,Int(x) - 9\,Int(x^2)$
$$= 4x \quad + 6\left(\frac{x^2}{2}\right) - 9\left(\frac{x^3}{3}\right) \text{ from ②}$$
$$= 4x \quad + 3x^2 - 9x^3$$

Third, by matrix multiplication:
$$Int(4+6x-9x^2) = Int\begin{pmatrix}4\\6\\-9\end{pmatrix}_{B_1}$$
$$= \underbrace{\begin{pmatrix}0&0&0\\1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{pmatrix}}_{[Int]_{B_1,B_2}}\begin{pmatrix}4\\6\\-9\end{pmatrix}_{B_1} = \begin{pmatrix}0\\4\\3\\-3\end{pmatrix}_{B_2} = 4x+3x^2-3x^3$$

So we can combine our notion of coordinates with the generalization $T: \mathbb{R}^n \to \mathbb{R}^m$ to represent any transformation:

Def: Let $T: V \to W$ be a linear transformation and $B_1 = \{v_1, \dots, v_n\}$, $B_2 = \{w_1, \dots, w_m\}$ be bases for V & W respectively. Let a_{ij}, $1 \le i \le m$, $1 \le j \le n$ be the scalars that determine T wrt B_1 & B_2. Then the $m \times n$ matrix (a_{ij}) is the matrix of T with respect to B_1 & B_2, $[T]_{B_1,B_2}$

Steps to find $[T]_{B_1,B_2}$: ① Find the image of every basis vector for V under T
② Find the coordinates of the image with respect to W's basis
③ Columns of $[T]_{B_1,B_2}$ = coordinates from ②