Question: What polynomial has $rep_{B_3} = \begin{pmatrix}-3\\1\\0\end{pmatrix}_{B_3} + \underbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}_{B_3}}_{u} = \begin{pmatrix}-2\\3\\3\end{pmatrix}_{B_3}$?

$$\begin{pmatrix}1&4&2\\1&1&-1\\-1&0&1\end{pmatrix}_{B_3\to B_1}\begin{pmatrix}-2\\3\\3\end{pmatrix}_{B_3} = \begin{pmatrix}16\\-2\\5\end{pmatrix}_{B_1} = 16-2x+5x^2$$

Notice: $1-2x+3x^2+15+2x^2$
$= 16-2x+5x^2$!

and
$$\begin{pmatrix}1&4&2\\1&1&-1\\-1&0&1\end{pmatrix}_{B_3\to B_1}\begin{pmatrix}1\\2\\3\end{pmatrix}_{B_3} = \begin{pmatrix}15\\0\\2\end{pmatrix}_{B_1} = 15 \quad +2x^2$$

So $A(u+v) = Au + Av$.

We like when this happens. Changing bases is an example of a linear map (or transformation) from a vector space to itself. More generally, we'd like maps from one vector space to any other:

Def: A linear map (or transformation) from V to W is a function $T: V \to W$ with the following properties:
① additivity: $T(u+v) = Tu + Tv \quad \forall u,v \in V$
② homogeneity: $T(\lambda v) = \lambda(Tv) \quad \forall \lambda \in F, v \in V$

We call $\mathcal{L}(V,W)$ the set of all linear maps from V to W.

Ex: Multiplication by $x^2 = T$. What are our options for V? for W?
Check the two properties:
① Let $p,q \in V$. Then $T(p+q) = x^2(p+q) = x^2p + x^2q = Tp + Tq$ ✓
② Let $p \in V$, $\lambda \in F$. Then $T(\lambda p) = x^2(\lambda p) = \lambda(x^2 p) = \lambda(Tp)$ ✓

(Non) Ex: $V = \mathbb{R}^2 = W$. Define $T: V \to W$ by $T(x_1,x_2) = (x_1^2, x_2^2)$. Is T linear?
Solution: No! $T((x_1,x_2)+(y_1,y_2)) = T((x_1+y_1, x_2+y_2))$
$$= ((x_1+y_1)^2, (x_2+y_2)^2) \neq (x_1^2+y_1^2, x_2^2+y_2^2)$$
homogeneity also fails.

Ex: zero map → Is $0 \in \mathcal{L}(V,W)$? $\forall v \in V$, $\underset{\substack{\uparrow\\ 0\text{ map}}}{0} \cdot v = \underset{\substack{\uparrow\\ 0\text{ vector in } W.}}{0}$
① $v_1, v_2 \in V$. Then $0(v_1+v_2) = 0 = 0+0 = 0\cdot v_1 + 0 \cdot v_2$ ✓
② $v \in V$, $\lambda \in F$. Then $0(\lambda v) = 0 = \lambda \cdot 0 = \lambda(0v)$ ✓

Ex (LOYO) Check that the map $T: \mathbb{R}_3 \to \mathbb{R}_3$ defined by $T(x,y,z) = (x+4y+2z, x+y-z, -x+z)$ is linear.

Solution: ① Let $(x_1,y_1,z_1), (x_2,y_2,z_2) \in \mathbb{R}^3$.
$$T((x_1,y_1,z_1)+(x_2,y_2,z_2)) = T(x_1+x_2, y_1+y_2, z_1+z_2)$$
$$= (x_1+x_2+4(y_1+y_2)+2(z_1+z_2), x_1+x_2+y_1+y_2-(z_1+z_2), -(x_1+x_2)+z_1+z_2)$$
$$= (x_1+4y_1+2z_1, x_1+y_1-z_1, -x_1+z_1) + (x_2+4y_2+2z_2, x_2+y_2-z_2, -x_2+z_2) = T(x_1,y_1,z_1) + T(x_2,y_2,z_2)$$

② Let $\lambda \in F \to \mathbb{R}$, $(x,y,z) \in \mathbb{R}^3$

$$T(\lambda(x,y,z)) = T((\lambda x, \lambda y, \lambda z)) = (\lambda x + 4(\lambda y) + 2(\lambda z),\ \lambda x + \lambda y - \lambda z,\ -\lambda x + \lambda z)$$
$$= \lambda(x+4y+2z,\ x+y-z,\ -x+z)$$
$$= \lambda T(x,y,z) \checkmark$$

Does this look familiar? Can we write it another way?

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{input into } T} = \underbrace{\begin{pmatrix} x + 4y + 2z \\ x + y - z \\ -x \quad + z \end{pmatrix}}_{\text{output / range}}$$

In general, any linear map from F^n to F^m is of the form

$$T(x_1, \dots, x_n) = (a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n,\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n, \dots, a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n)$$

$$\Rightarrow T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$