Interlude on Matrix Multiplication

For You: Find the dot product of the following vectors (if possible):

$(-1, 1)\cdot(2,3)$ $\quad$ $(-1,0,1)\cdot(2,3)$ $\quad$ $(6,-4,2,1)\cdot(-1,2,-3,4)$

Solutions: $-1(2)+1(3) = \boxed{1}$

Not possible! Must be the same size

$-6+-8+-6+4=-16$

If you know dot product, you know matrix multiplication! It's just a whole bunch of dot products:

Ex:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -6 & -5 \\ -4 & -3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -6-8-6 & -5-6-3 \\ -24-20-12 & -20-15-6 \end{pmatrix} = \begin{pmatrix} -20 & -14 \\ -56 & -41 \end{pmatrix}$$

$\underbrace{2\times 3 \quad 3\times 2}_{\text{match}}$ $\qquad$ 2×2 result

In General, if $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{m\times p}$, we can find the product $AB := (AB)_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$, which is an $n\times p$ matrix.

You try: Multiply the following matrices:

$$\begin{pmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -4 & -3 & -2 & -1 \\ 0 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 7 & 2 & -3 \\ 4 & 3 & 2 & 1 \\ -4 & -1 & 2 & 5 \end{pmatrix}$$

$3\times 2 \quad 2\times 4 \qquad 3\times 4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 0 & -1 \\ 3 & -6 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 27 & -3 \end{pmatrix}$$

$1\times 4 \quad 4\times 2 \qquad 1\times 2$

$$\begin{pmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -6 & -5 \\ -4 & -3 \\ -2 & -1 \end{pmatrix}$$

X 3x2 and 3x2 cannot be multiplied.
(Can they be added? Subtracted? Scalar multiplied?)

$$\begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -25 \\ -4 \end{pmatrix}$$

$3\times 3 \quad 3\times 1 \qquad 3\times 1$

Def The change of basis matrix, or transition matrix from a basis B_1 to another basis B_2 is the $n \times n$ matrix (where $\dim V = n$)

$$\begin{pmatrix} | & | & & | \\ rep_{B_2}(u_1) & rep_{B_2}(u_2) & \cdots & rep_{B_2}(u_n) \\ | & | & & | \end{pmatrix}_{B_1 \to B_2}$$

where $B_1 = \{u_1, \ldots, u_n\}$ and each column corresponds to u_i's coordinates wrt B_2.

In the context of matrix multiplication, left multiplication by this matrix converts a representation of a vector with respect to B_1 to its representation with respect to B_2.

In our example,

COB matrix is $\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}_{B_1 \to B_2}$ Then $\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}_{B_1 \to B_2} \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{B_1} = \begin{pmatrix} \frac{8}{5} \\ -\frac{7}{5} \end{pmatrix}_{B_2}$ ✓

Quick aside: We solved a system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

by reducing $\left(\begin{array}{ccc|c} a_{11} & a_{12} \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} & b_m \end{array}\right)$ and read off solution.

We can also think of this as a matrix equation using matrix multiplication:

$Ax = b$ where

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

$$m \times n \quad n \times 1 \quad m \times 1$$

Another COB ex: We've shown that $B_1 = \{1, x, x^2\}$ and $B_3 = \{1+x-x^2, 4+x, 2-x+x^2\}$ are both bases for P_2. Then the COB matrix from B_3 to B_1 is

$$\left(rep_{B_1}(1+x-x^2) \,\middle|\, rep_{B_1}(4+x) \,\middle|\, rep_{B_1}(2-x+x^2) \right) = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}_{B_3 \to B_1} = A$$

We had $rep_{B_3}(1-2x+3x^2) = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}_{B_3} = v$

Then $\begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}_{B_3 \to B_1} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}_{B_3} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}_{B_1}$ ✓

Question: What polynomial has $rep_{B_3} = 4 \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}_{B_3} = \begin{pmatrix} -12 \\ 4 \\ 0 \end{pmatrix}_{B_3}$?

$\begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}_{B_3 \to B_1} \begin{pmatrix} -12 \\ 4 \\ 0 \end{pmatrix}_{B_3} = \begin{pmatrix} 4 \\ -8 \\ 12 \end{pmatrix}_{B_1}$ which is $4 - 8x + 12x^2 = 4(1-2x+3x^2)$.

$\Rightarrow A(4v) = 4(Av)$!