① Write $\begin{pmatrix}3\\2\end{pmatrix}$ in both bases.

<u>Solution</u>: $rep_{B_1}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}_{B_1}$ and to find $rep_{B_2}\begin{pmatrix}3\\2\end{pmatrix}$, we have $\begin{pmatrix}3\\2\end{pmatrix} = a\begin{pmatrix}1\\3\end{pmatrix} + b\begin{pmatrix}-1\\2\end{pmatrix}$

$$rep_{B_2}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}\frac{8}{5}\\ -\frac{7}{5}\end{pmatrix}_{B_2} \Longleftarrow \left\{\begin{array}{l} 3 = a-b \rightarrow a = b+3 \\ 2 = 3a+2b \\ 2 = 3(b+3)+2b \\ -7 = 5b \quad b = -\frac{7}{5} \\ \rightarrow a = \frac{8}{5}\end{array}\right.$$

② Write $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$ $(B_1)$ in terms of $B_2$.

<u>Solution</u>: $\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}1\\3\end{pmatrix} + b\begin{pmatrix}-1\\2\end{pmatrix} \Rightarrow$ $1 = a-b \rightarrow a = b+1$, $0 = 3a+2b$

$0 = 3(b+1)+2b \rightarrow -3 = 5b \rightarrow b = -\frac{3}{5}$, $a = \frac{2}{5}$

$$rep_{B_2}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{2}{5}\\ -\frac{3}{5}\end{pmatrix}_{B_2}$$

$\begin{pmatrix}0\\1\end{pmatrix} = a\begin{pmatrix}1\\3\end{pmatrix} + b\begin{pmatrix}-1\\2\end{pmatrix} \rightarrow$ $0 = a-b \longrightarrow a = b$; $1 = 3a+2b \longrightarrow 1 = 5b \quad b = \frac{1}{5}, a = \frac{1}{5}$

$$rep_{B_2}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{5}\\ \frac{1}{5}\end{pmatrix}_{B_2}$$

Notice: $rep_{B_1}\begin{pmatrix}3\\2\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix}_{B_1} + 2\begin{pmatrix}0\\1\end{pmatrix}_{B_1} = \begin{pmatrix}3\\2\end{pmatrix}$.

Is the same true when we write $B_1$ in terms of $B_2$?

That is, is

$$rep_{B_2}\begin{pmatrix}3\\2\end{pmatrix} = 3\begin{pmatrix}\frac{2}{5}\\ -\frac{3}{5}\end{pmatrix}_{B_2} + 2\begin{pmatrix}\frac{1}{5}\\ \frac{1}{5}\end{pmatrix}_{B_2} ?$$

Let's see this a little more generally:

If we have two bases for a vector space, $B_1 = \{u_1, u_2, \ldots, u_n\}$ and $B_2 = \{v_1, v_2, \ldots, v_n\}$, we can write each vector in $B_1$ as a linear combination of the vectors in $B_2$ (coordinates) or vice versa:

$$\begin{aligned} u_1 &= a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \\ u_2 &= b_1 v_1 + b_2 v_2 + \ldots + b_n v_n \\ &\vdots \\ u_n &= k_1 v_1 + k_2 v_2 + \ldots + k_n v_n \end{aligned}$$

Then if we know a vector's representation with respect to the first basis, we are able to find its representation with respect to the second basis!

$$\begin{aligned} v &= \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n \\ &= \lambda_1(a_1 v_1 + \ldots + a_n v_n) + \lambda_2(b_1 v_1 + \ldots + b_n v_n) + \ldots + \lambda_n(k_1 v_1 + \ldots + k_n v_n) \\ &= (\lambda_1 a_1 + \lambda_2 b_1 + \ldots + \lambda_n k_1) v_1 + \ldots + (\lambda_1 a_n + \lambda_2 b_n + \ldots + \lambda_n k_n) v_n \\ &= (\lambda_1\ \lambda_2 \ldots \lambda_n)\begin{pmatrix}a_1\\b_1\\ \vdots\\k_1\end{pmatrix} v_1 + \ldots + (\lambda_1\ \lambda_2 \ldots \lambda_n)\begin{pmatrix}a_n\\b_n\\ \vdots\\k_n\end{pmatrix} v_n \\ &= (\lambda_1\ \lambda_2 \ldots \lambda_n)\begin{pmatrix}a_1 & a_2 & \cdots & a_n\\ b_1 & b_2 & \cdots & b_n\\ \vdots & & \ddots & \vdots \\ k_1 & k_2 & \cdots & k_n\end{pmatrix}\begin{pmatrix}v_1\\ \vdots \\ \vdots \\ v_n\end{pmatrix} \end{aligned}$$

BOOK Hefferon $= \begin{pmatrix}a_1 & b_1 & \cdots & k_1\\ a_2 & b_2 & \cdots & k_2\\ \vdots & & \ddots & \\ a_n & b_n & \cdots & k_n\end{pmatrix}\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix}(v_1 \ldots v_n)$

Def: The change of basis matrix, or transition matrix from a basis $B_1$ to another basis $B_2$ is the $n \times n$ matrix (where $\dim V = n$)

$$\begin{pmatrix} | & | & & | \\ \mathrm{rep}_{B_2}(u_1) & \mathrm{rep}_{B_2}(u_2) & \cdots & \mathrm{rep}_{B_2}(u_n) \\ | & | & & | \end{pmatrix}_{B_1 \to B_2}$$

where $B_1 = \{u_1, \ldots, u_n\}$ and each column corresponds to $u_i$'s coordinates wrt $B_2$.

In the context of matrix multiplication, left multiplication by this matrix converts a representation of a vector with respect to $B_1$ to its representation with respect to $B_2$.

In our example,

COB matrix is $\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}_{B_1 \to B_2}$ Then $\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}_{B_1 \to B_2} \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{B_1} = \begin{pmatrix} \frac{8}{5} \\ -\frac{7}{5} \end{pmatrix}_{B_2}$ ✓