Another example of "vector" $\longrightarrow$ vector:

Ex: Find a basis for M_2 which contains $\begin{pmatrix}1&0\\1&0\end{pmatrix}$ and $\begin{pmatrix}-1&1\\-1&0\end{pmatrix}$

Solution: We know $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is a basis for M_2, so it spans M_2.

Then $\left\{\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}-1&1\\-1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ also spans M_2.

We need to keep, but there's too many vectors for a basis, so we need to find dependencies.

$$a_1\begin{pmatrix}1&0\\1&0\end{pmatrix}+a_2\begin{pmatrix}-1&1\\-1&0\end{pmatrix}+a_3\begin{pmatrix}1&0\\0&0\end{pmatrix}+a_4\begin{pmatrix}0&1\\0&0\end{pmatrix}+a_5\begin{pmatrix}0&0\\1&0\end{pmatrix}+a_6\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\begin{aligned}a_1-a_2+a_3 &= 0\\ a_2+a_4 &= 0\\ a_1-a_2+a_5 &= 0\\ a_6 &= 0\end{aligned}$$

OR, straight to here "vectors"

$$\xrightarrow{\text{Matrix}}\begin{pmatrix}1&-1&1&0&0&0\\0&1&0&1&0&0\\1&-1&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\xrightarrow{-R_1+R_3}$$

$$\begin{pmatrix}1&-1&1&0&0&0\\0&1&0&1&0&0\\0&0&-1&0&1&0\\0&0&0&0&0&1\end{pmatrix}\xrightarrow{-R_3}\begin{pmatrix}\boxed{1}&-1&1&0&0&0\\0&\boxed{1}&0&1&0&0\\0&0&\boxed{1}&0&-1&0\\0&0&0&0&0&\boxed{1}\end{pmatrix}$$

$\Rightarrow \left\{\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}-1&1\\-1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is a basis for M_2.

! There may be others, OR other ways to find this

We have a name for this "vector" $\to$ vector thing $\to$ coordinates

Def In a vector space with basis $B=\{v_1,\dots,v_n\}$, the representation of v with respect to B is the column vector of coefficients of a vector v when it is written as a linear combination of $v_1,\dots,v_n$. That is,

$$\text{rep}_B(v)=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}_B \quad \text{where } v=c_1v_1+c_2v_2+\dots+c_nv_n$$

We call the c_i the coordinates of v with respect to B.

For example, M_2 has basis $B=\left\{\underset{E_{11}}{\begin{pmatrix}1&0\\0&0\end{pmatrix}},\underset{E_{12}}{\begin{pmatrix}0&1\\0&0\end{pmatrix}},\underset{E_{21}}{\begin{pmatrix}0&0\\1&0\end{pmatrix}},\underset{E_{22}}{\begin{pmatrix}0&0\\0&1\end{pmatrix}}\right\}$

So $\begin{pmatrix}1&0\\1&0\end{pmatrix}=1\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\begin{pmatrix}0&1\\0&0\end{pmatrix}+1\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$\Rightarrow \text{rep}_B\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix}_B$

Ex In P_2, we have bases $B_1=\{1,x,x^2\}$, $B_2=\{x^2,x,1\}$, $B_3=\{1+x-x^2,4+x,2-x+x^2\}$ (from last class)

Write $1-2x+3x^2$ in terms of each basis to find its coordinates wrt each.

$\underline{B_1}$: $1-2x+3x^2 = \underline{1}\ 1 + \underline{-2}\ x + \underline{3}\ x^2 \Rightarrow \operatorname{rep}_{B_1}(1-2x+3x^2) = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}_{B_1}$

$\underline{B_2}$: $1-2x+3x^2 = \underline{3}\ x^2 + \underline{-2}\ x + \underline{1}\ 1 \Rightarrow \operatorname{rep}_{B_2}(1-2x+3x^2) = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}_{B_2}$

not the same! We need the order in basis

$\underline{B_3}$: $1-2x+3x^2 = \underline{a}(1+x-x^2) + \underline{b}(4+x) + \underline{c}(2-x+x^2)$

On your own: Solve for a, b, c.

$$\begin{aligned} 1 &= a + 4b + 2c \\ -2 &= a + b - c \\ 3 &= -a + c \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 1 & 1 & -1 & -2 \\ -1 & 0 & 1 & 3 \end{array}\right) \xrightarrow{G\text{-}J} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \begin{aligned} a &= -3 \\ b &= 1 \\ c &= 0 \end{aligned}$$

$$\operatorname{rep}_{B_3}(1-2x+3x^2) = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}_{B_3}$$

Pause → Big Picture using coordinates, we can simplify computations:

V an arbitrary, finite-dimensional vector space ($\dim = n$) over $\mathbb{R}$	$\longleftrightarrow$	$\mathbb{R}^n$
basis $B = \{v_1, \dots, v_n\}$	$\longleftrightarrow$	$\{e_1, \dots, e_n\}$, $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$
arbitrary vector $v = c_1v_1 + \dots + c_nv_n$	$\longleftrightarrow$	$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$
To find out about relationships over here!	$\longleftarrow$	We can use all of our matrix tricks for bases, linear independence, solution sets, etc. over here

Ex Find a basis for P_3 containing the linearly independent set

$S = \{x^3 - 2,\ x^2 + 3x\}$

Solution: Use the same idea from M_2 example. Expand by adding a full spanning set, then trim down.

$S' = \{x^3 - 2,\ x^2 + 3x,\ x^3,\ x^2,\ x,\ 1\}$ spans P_3.

coordinates in terms of Basis, $B = \{x^3, x^2, x, 1\}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(Thank goodness for computers!)

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{G\text{-}J} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & 0 \end{pmatrix}$$

Then a basis is $\{x^3 - 2,\ x^2 + 3x,\ x^3,\ x^2\}$.

(Note: what do the free variables tell you?)

Coordinates can be super helpful when working with vector spaces of polynomials, but can be confusing when working in $\mathbb{R}^n$:

For example, $B_1 = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and $B_2 = \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}$ are both bases for $V = \mathbb{R}^2$