$$\left(\begin{array}{cccc|c} 0 & 2 & -4 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & 5 & -11 & -1 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right) \xrightarrow{G\text{-}J} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

not needed · free variable · sec?

$\to a_1 - a_3 = 0 \to a_1 = a_3$

$a_2 - 2a_3 = 0 \to a_2 = 2a_3$

$a_4 = 0$

Set $a_3 = 1 \Rightarrow a_1 = 1, a_2 = 2$

$\Rightarrow v_1 + 2v_2 + v_3 = 0 \to v_3 = -v_1 - 2v_2 \therefore v_3$ is extra.

★ Keep the vectors with pivots → that's our basis!

$\text{span}(v_1, v_2, v_4) = \text{span}(v_1, v_2, v_3, v_4)$.

On your own: $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$.

Questions we can ask:
1. Are these linearly independent
2. Find a smaller set that also spans span(S).
3. Find a basis for span(S).

How do you answer these? →

So far, we've used matrices to tell us about our vectors or solutions to a system of equations. We can also learn from a matrix itself:

Def: The column space of a matrix is the vector space spanned by the columns of the matrix.

To find a basis for the column space, row reduce the matrix into REF or RREF & choose the original columns corresponding to pivots.

$\Rightarrow$ If A is a matrix, $\text{rank}(A) = \#$ pivots of REF or RREF of A
$= $ dimension of column space of A.

Ex: Find a basis for the column space of $A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{4}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

So: $\left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -4 \end{pmatrix} \right\}$ is a basis for the column space.

$\text{rank}(A) = 3 = \dim(\text{col}(A))$.

Ex: Find a basis for the column space of $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis

this is the same answer as

"Find a basis for the subspace spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} \right\}$.

Def: The nullspace of a matrix is the solution set to the homogeneous system of equations for which A is the coefficient matrix. The dimension of the nullspace of A is called its nullity.

Ex: Find a basis for the nullspace of $A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}$ (above).

i.e. Find solutions to

$$\begin{aligned} 2x_1 \quad + 3x_3 + 4x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0 \\ 3x_1 + x_2 \quad + 2x_4 &= 0 \\ x_1 \quad - 4x_3 - x_4 &= 0 \end{aligned}$$

We use RREF from above, though this time it answers a different question.

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \to x_1 = -\frac{13}{11}x_4 \\ \to x_2 = \frac{17}{11}x_4 \\ \to x_3 = -\frac{6}{11}x_4 \end{array}$$

Then the solution set is

$$\left\{ \begin{pmatrix} -\frac{13}{11}s \\ \frac{17}{11}s \\ -\frac{6}{11}s \\ s \end{pmatrix} \middle| s \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} -13 \\ 17 \\ -6 \\ 11 \end{pmatrix} \right\}$$

= Nullspace of A.

$\Rightarrow$ Nullity of A is 1.

Thm: Let A be a matrix. Then rank(A) + nullity(A) = # columns of A.

We'll come back to these ideas. First, let's revisit bases:

Ex: Find a basis for the vector space $V = \{p \in P_3 \mid p(2) = p(3)\}$ (Note: you may want to verify this IS a v.s.)

Strategy: Find what a generic element in V looks like, then set each parameter to 1 and rest 0 to form basis (Note: you may want to think about why this will always give you a basis)

We know a generic element from P_3 looks like

$p(x) = ax^3 + bx^2 + cx + d$

and if $p \in V$, we have $8a + 4b + 2c + d = 27a + 9b + 3c + d$

$\Rightarrow 19a + 5b + c = 0$ so $c = -19a - 5b$.

Then a generic element of V looks like $ax^3 + bx^2 + (-19a - 5b)x + d$.

Set each free variable to 1 & rest 0:

$\{\underbrace{x^3 - 19x}_{a=1}, \underbrace{x^2 - 5x}_{b=1}, \underbrace{1}_{d=1}\}$ is a basis for V. ($\dim(V) = 3$)

Can we use matrices instead? This one was simple (one equation), but matrices can help with more complicated polynomial (or non $\mathbb{R}^n$) examples.

Ex: Find a basis for $\text{span}(1+x-x^2, 4+x, 2-x+x^2, x-5x^2)$

Thought Process: These live in P_2, but $\dim(P_2) = 3$ so there must be redundant vectors. ALWAYS CHECK FOR REDUNDANCIES (even if it only had 3 vectors)

Find solutions to $a_1(1+x-x^2) + a_2(4+x) + a_3(2-x+x^2) + a_4(x-5x^2) = 0$.

Collect like terms:

3 equations, 4 unknowns more complicated $\to$ matrix!

constants: $a_1 + 4a_2 + 2a_3 = 0$

x: $x(a_1 + a_2 - a_3 + a_4) = 0$

x^2: $x^2(-a_1 + a_3 - 5a_4) = 0$

$$\begin{pmatrix} 1 & 4 & 2 & 0 \\ 1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -5 \end{pmatrix} \xrightarrow{\text{G-J}} \begin{pmatrix} 1 & 0 & 0 & 24/3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 11/3 \end{pmatrix}$$

$\uparrow \uparrow \uparrow$ pivots $\qquad \uparrow$ free variable.

Choose columns with pivots:

$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$. NOT my basis! our vectors are polynomials.

Basis $= \{\underset{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{1 + x - x^2}, \underset{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}}{4 + x}, \underset{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{2 - x + x^2}\}$. What is this space?

"vector" $\to$ vector

Another example of "vector" $\longrightarrow$ vector:

Ex: Find a basis for M_2 which contains $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

Solution: We know $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is a basis for M_2, so it spans M_2.

Then $\left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ also spans M_2.

We need to keep $\nwarrow \uparrow$, but there's too many vectors for a basis, so we need to find dependencies.