Def: In each row of a system, the first variable with a nonzero coefficient is the row's leading variable and will become the pivot (leading 1s)

The number of pivots in REF (or RREF) tells us about the solution set to the system of equations. Can we determine a rule?

① If each variable has a pivot (a leading 1 in each column of the coefficient matrix), the system has a unique solution.

② If there's a pivot in the RHS, the system has no solution. (Why?)

③ Otherwise, the system has infinitely many solutions → given by parameters.

Def: Variables that correspond to columns with no pivot are called free variables or parameters

Def: The number of pivots in REF is called the rank of a matrix.

On your own: If you have a system of m equations and n variables, what are the possibilities if $m<n$? If $m>n$? If $m=n$?

Solving systems of equations is one reason we look at matrices, but not the only use:

Ex: $v_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} -6 \\ 3 \\ 7 \end{pmatrix}$ $v_3 = \begin{pmatrix} -9 \\ 3 \\ k \end{pmatrix}$ When is this set a basis for $\mathbb{R}^3$?

Same question as:

① When is v_3 NOT a linear combination of v_1 and v_2?

It is when:

$$a\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + b\begin{pmatrix} -6 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \\ k \end{pmatrix}$$ 3 eqns, 2 unkn

$$a\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + b\begin{pmatrix} -6 \\ 3 \\ 7 \end{pmatrix} + c\begin{pmatrix} -9 \\ 3 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

• Solve for a, b, c

② When are v_1, v_2, v_3 linearly independent?

③ When do these three span $\mathbb{R}^3$?

This is on WeBWorK. For all 3, set up $\begin{pmatrix} 3 & -4 & -9 \\ -2 & 3 & 3 \\ 0 & 7 & k \end{pmatrix}$, find REF or RREF and determine any restrictions on k.

Ex: Or, our question from before:

Find a basis for the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ -11 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$

↳ i.e. Find a linearly independent subset.

Does $a_1\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 2 \\ -1 \\ 5 \\ 3 \end{pmatrix} + a_3\begin{pmatrix} -4 \\ 1 \\ -11 \\ -6 \end{pmatrix} + a_4\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ have nontrivial solutions?

4 equations, 4 unknowns. Matrix!

$$\left(\begin{array}{cccc|c} 0 & 2 & -4 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & 5 & -11 & -1 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right) \xrightarrow{\text{G-J}} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(last column of the first matrix: ↑ not needed; third column of the reduced matrix: ↑ free variable; last column of the reduced matrix: ↑ see?)

$\rightarrow a_1 - a_3 = 0 \rightarrow a_1 = a_3$

$a_2 - 2a_3 = 0 \rightarrow a_2 = 2a_3$

$a_4 = 0$

Set $a_3 = 1 \Rightarrow a_1 = 1, a_2 = 2$

$\Rightarrow v_1 + 2v_2 + v_3 = 0 \rightarrow v_3 = -v_1 - 2v_2$ ∴ v_3 is extra.

★ Keep the vectors with pivots → that's our basis!

$\text{Span}(v_1, v_2, v_4) = \text{span}(v_1, v_2, v_3, v_4)$.

On your own: $S = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix} \right\}$.

How do you answer these? →

Questions we can ask:
1. Are these linearly independent
2. Find a smaller set that also spans span(S).
3. Find a basis for span(S).

So far, we've used matrices to tell us about our vectors or solutions to a system of equations. We can also learn from a matrix itself:

Def: The column space of a matrix is the vector space spanned by the columns of the matrix.

To find a basis for the column space, row reduce the matrix into REF or RREF & choose the original columns corresponding to pivots.

$\Rightarrow$ If A is a matrix, rank(A) = # pivots of REF or RREF of A
= dimension of column space of A.

Ex: Find a basis for the column space of $A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 6 & -4 & -1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(↑ ↑ ↑ under the first three columns of each matrix)

So: $\left\{ \begin{pmatrix}2\\0\\3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\\-4\end{pmatrix} \right\}$ is a basis for the column space.

$\text{rank}(A) = 3 = \dim(\text{col}(A))$.