OR, continue to RREF, which is Gauss-Jordan Elimination:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{R_1+3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & -4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow[R_2-R_3]{R_1-5R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -24 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{array}\right) \rightarrow \text{Now read off the values.}$$

* I will not make you go through this entire process on a test (since we have computers to do this), but you should understand the process in case I ask you to do one step, or check if it's in RREF *
(or something similar)

More importantly, you need to be able to interpret the final matrix.

$$\begin{bmatrix} 0 & 1 & -6 & 0 \\ 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$ $\leftrightarrows$ to fix

not in echelon form

$$\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced row echelon

$$\begin{bmatrix} 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

reduced row echelon form (0's above & below 1s)

$$\begin{bmatrix} 7 & -4 & -5 & 8 & -4 \\ 0 & -7 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -10 \\ 0 & 6 & 0 & 1 & 0 \end{bmatrix}$$

echelon form

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

not in echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

row echelon form (0s below 1s)

Solve the systems.

Ex 1

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 3 \\ -2x_1 - x_2 \quad\quad &= -4 \\ 4x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ -2 & -1 & 0 & -4 \\ 4 & 2 & 3 & 7 \end{array}\right)$$

G-J Maple, or Mathematica, etc.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)$$

Read off:

$$x_1 = 2$$
$$x_2 = 0$$
$$x_3 = -\frac{1}{3}$$

This system has a unique solution.

Ex 2

$$\begin{aligned} x - y + z &= 5 \\ x - y \quad &= 2 \\ 2x - 2y + z &= 7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & -1 & 0 & 2 \\ 2 & -2 & 1 & 7 \end{array}\right)$$

G-J

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Read off:

$$x - y = 2$$
$$z = 3$$

This system has infinitely many solutions:

$$\left\{ \begin{pmatrix} y+2 \\ y \\ 3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \mathbb{R} \right\}$$

Ex 3

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_1 + x_2 + 3x_3 &= 5 \\ 3x_1 + 2x_2 + 4x_3 &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 3 & 5 \\ 3 & 2 & 4 & 9 \end{array}\right)$$

G-J

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Read off:

$$x + 2z = 0$$
$$y - z = 0$$
$$0 = 1$$

ò!

This system has no solution since $0 \neq 1$.