Lemma proof idea: Say $\dim(V)=n \Rightarrow$ spanning set has size $\geq n$.
from (◎) Thm, Linearly indep. set $\leq n$ ✓

Proof of Thm: Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases for V.
Then $\mathcal{B}_1$ is a set of linearly independent vectors & $\mathcal{B}_2$ is a spanning set.
By the Lemma, $|\mathcal{B}_1| \leq |\mathcal{B}_2|$. ①
Also, $\mathcal{B}_2$ is a set of Lin. indep vectors and $\mathcal{B}_1$ is a spanning set
$\Rightarrow |\mathcal{B}_2| \leq |\mathcal{B}_1|$ ② $\xrightarrow{①+②} |\mathcal{B}_1| = |\mathcal{B}_2|$ ∎

Thm: If U is a subspace of V then $\dim(U) \leq \dim(V)$. If $\dim(U)=\dim(V)$, then $U=V$.

Two useful ideas: (If $\dim V = n$)
① any collection of n linearly independent vectors is a basis. (i.e. span V).
② any spanning set of size n is a basis (i.e. must be linearly independent)

Ex: What are possible subspaces of $\mathbb{R}^3$?
$\dim(\mathbb{R}^3)=3$ so by Thm,

dim	Basis	vectors in subspace	geometric description
0	$\emptyset$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	origin
1	$\begin{pmatrix}a\\b\\c\end{pmatrix}$	$k\begin{pmatrix}a\\b\\c\end{pmatrix}$	Line in direction $\langle a,b,c\rangle$ through origin
2	$\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}d\\e\\f\end{pmatrix}\right\}$ not $k\begin{pmatrix}a\\b\\c\end{pmatrix}$	$k_1\begin{pmatrix}a\\b\\c\end{pmatrix} + k_2\begin{pmatrix}d\\e\\f\end{pmatrix}$	plane through origin containing $\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}d\\e\\f\end{pmatrix}\right\}$
3	$\{e_1, e_2, e_3\}$	$\left\{\begin{pmatrix}a\\b\\c\end{pmatrix} \mid a,b,c \in \mathbb{R}\right\}$	$\mathbb{R}^3$

Ex: Find a basis for the subspace of $\mathbb{R}^4$ spanned by $\left\{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-1\\5\\3\end{pmatrix}, \begin{pmatrix}-4\\1\\-11\\-6\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}\right\}$

What can happen?
If it spans $\mathbb{R}^4$, we know it's already a basis by useful idea ②.
OR there could be dependency (redundant vectors → proper subspace of $\mathbb{R}^4$)
How do we determine?

$$a\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + b\begin{pmatrix}2\\-1\\5\\3\end{pmatrix} + c\begin{pmatrix}-4\\1\\-11\\-6\end{pmatrix} + d\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \begin{cases} 2b - 4c + d = 0 \\ a - b + c = 0 \\ a + 5b - 11c - d = 0 \\ 3b - 6c = 0 \end{cases}$$ ← yuck.

There's another way!

Systems of Linear Equations

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$

has coefficient matrix
$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{mn} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

and Augmented matrix
$$\bar{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{mn} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Thinking back to high school, what can we do to a system of equations without changing the solution set? (To make it easier to solve)

The following "row operations" don't change the set of solutions to the system:

① Replace row j with (row j + multiple of some other row)

$$R_j \to R_j + cR_i$$

(this is the same as elimination in high school → $\begin{matrix} x+y=7 \\ 3x+4y=12 \end{matrix} \xrightarrow{R_2+(-3)R_1} \begin{matrix} x+y=7 \\ y=-9 \end{matrix}$)

② Multiply a row by a constant

$$R_j \to kR_j$$

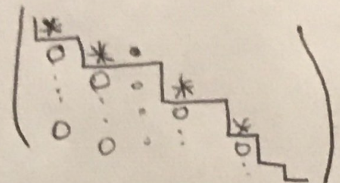
Note: we can see in one variable easily why this doesn't change solution:
$3x=9$ has same solution as $\frac{1}{3}(3x=9)$

③ Swap two rows.

$$R_i \leftrightarrow R_j$$

(We didn't really do this in HS, but it's clear it does not change the solution set).

We use these row operations to eliminate variables and get a matrix of the form



Either row echelon form (REF) or reduced row echelon form (RREF)

Row Echelon Form → $*$s are 1^s and 0^s under each.

Reduced Row Echelon Form → same as above, but 0^s above leading 1^s as well.

The definitions are technical, so let's see through example:

Ex: Solve the system
$$\begin{aligned} 3x_2+4x_3&=-5 \\ 3x_1-7x_2+8x_3&=9 \\ 3x_1-9x_2+6x_3&=15 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 3 & 4 & -5 \\ 3 & -7 & 8 & 9 \\ 3 & -9 & 6 & 15 \end{array}\right).$$

Solution: First, create the augmented matrix ↗. We need a nonzero entry in the top left. Our goal is to have ⌐_, so we need a nonzero entry in the top left.

$$\left(\begin{array}{ccc|c} 0 & 3 & 4 & -5 \\ 3 & -7 & 8 & 9 \\ 3 & -9 & 6 & 15 \end{array}\right) \xrightarrow[③]{R_1\leftrightarrow R_3} \left(\begin{array}{ccc|c} 3 & -9 & 6 & 15 \\ 3 & -7 & 8 & 9 \\ 0 & 3 & 4 & -5 \end{array}\right) \xrightarrow[②]{\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 3 & -7 & 8 & 9 \\ 0 & 3 & 4 & -5 \end{array}\right) \xrightarrow[①]{R_2+(-3)R_1}$$

here, make all entries ↓ 0 below the 1 — "use the pivot 1"

Now, you could leave the 3 and work with it, but I find it's easier to make mistakes that way, so I'll change it to a 1.

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 2 & 2 & -6 \\ 0 & 3 & 4 & -5 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 4 & -5 \end{array}\right) \xrightarrow{R_3+(-3)R_2} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

↑ use the 1! If you try R_3+R_1, what happens?

Now continue to build your staircase.

We could stop here (REF) or continue to RREF

STOPPING is Gaussian Elimination with back substitution:

read $\boxed{x_3=4}$ (R_3) ⇒ $x_2+x_3=-3$ (R_2) ⇒ $\boxed{x_2=-7}$
$x_2+4=-3$

$x_1-3x_2+2x_3=5$ (R_1)
$x_1-3(-7)+2(4)=5$
$\boxed{x_1=-24}$

OR, continue to RREF, which is Gauss-Jordan Elimination:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{R_1+3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & -4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow[R_2-R_3]{R_1-5R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -24 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

→ Now read off the values.

* I will not make you go through this entire process on a test (since we have computers to do this), but you should understand the process in case I ask you to do one step, or check if it's in RREF * (or something similar)

More importantly, you need to be able to interpret the final matrix.