Warm up for you: $S = \{x^2-4,\ 2-x,\ x^2+x+2\}$

① Is S linearly independent or linearly dependent?

② What is $\operatorname{span}(S)$?

Solution: ① $a_1(x^2-4) + a_2(2-x) + a_3(x^2+x+2) = 0\cdot x^2 + 0\cdot x + 0$

$$\begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}\begin{cases} a_1 + a_3 = 0 \\ -a_2 + a_3 = 0 \\ -4a_1 + 2a_2 + 2a_3 = 0 \end{cases} \quad \begin{array}{l} \\ -E_2 \\ E_3 + 4E_1 \end{array} \Rightarrow \begin{cases} a_1 + a_3 = 0 \\ a_2 - a_3 = 0 \\ 2a_2 + 6a_3 = 0 \end{cases} \quad -2E_2 + E_3 \quad \begin{cases} a_1 + a_3 = 0 & a_1 = 0 \\ a_2 - a_3 = 0 \Rightarrow & a_2 = 0 \\ 8a_3 = 0 & a_3 = 0 \end{cases}$$

$\Rightarrow$ S is linearly independent.

② What $\alpha, \beta, \gamma \in \mathbb{R}^3$ can we have to guarantee existence of a_1, a_2, a_3?

$a_1(x^2-4) + a_2(2-x) + a_3(x^2+x+2) = \gamma x^2 + \beta x + \gamma$

$$\begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}\begin{cases} a_1 + a_3 = \alpha \\ -a_2 + a_3 = \beta \\ -4a_1 + 2a_2 + 2a_3 = \gamma \end{cases} \quad \begin{array}{l} \\ -E_2 \\ E_3 + 4E_1 \end{array} \begin{cases} a_1 + a_3 = \alpha \\ a_2 - a_3 = -\beta \\ 2a_2 + 6a_3 = \gamma + 4\alpha \end{cases} \quad -2E_2 + E_3 \begin{cases} a_1 + a_3 = \alpha \\ a_2 - a_3 = -\beta \\ \underbrace{8a_3 = \gamma + 4\alpha + 2\beta} \end{cases}$$

no matter what α, β, γ are, $\exists$ solution for a_1, a_2, a_3.

$\Rightarrow \operatorname{span}(S) = P_2$.

We call S a basis for P_2.

Def: A basis of V is a list of vectors in V that is linearly independent & spans V.

Thm: $\{v_1, \dots, v_n\} \subset V$ is a basis of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \dots + a_nv_n$ where $a_1, \dots, a_n \in F$.

Proof in Axler → Try it on your own before looking. Need both ways $\Rightarrow$ & $\Leftarrow$

Thm: The number of vectors in any basis for a vector space V is constant. (we will prove this).

Ex: We saw $S = \{x^2-4,\ 2-x,\ x^2+x+2\}$ is a basis for P_2. What else is?

Basis = set of efficient building blocks

not too many, Lin. Indep & spans the space.

$\mathcal{B} = \{1, x, x^2\}$ is the standard basis for P_2.

Def: The dimension of a vector space is the number of vectors in any basis for the vector space

Ex: $\dim(P_2) = 3$. $\quad \dim(P_n) = n+1$

Ex: $\mathbb{R}^4$ has basis $\left\{ \underset{e_1}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}, \underset{e_2}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}, \underset{e_3}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}, \underset{e_4}{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}} \right\}$ called the standard basis.

$\Rightarrow \dim(\mathbb{R}^4) = 4$.

$\dim(\mathbb{R}^n) = n$, basis $= \{e_1, e_2, \dots, e_n\}$

Lemma: In a vector space V, the size of any set of linearly independent vectors is less than or equal to the size of any spanning set. → Use to prove Thm.

Lemma proof idea: Say $\dim(V) = n \Rightarrow$ spanning set has size $\geq n$.
from ⓪ Thm, Linearly indep. set $\leq n$ ✓

Proof of Thm: Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases for V.
Then $\mathcal{B}_1$ is a set of linearly independent vectors & $\mathcal{B}_2$ is a spanning set.
By the Lemma, $|\mathcal{B}_1| \leq |\mathcal{B}_2|$. ①
Also, $\mathcal{B}_2$ is a set of Lin. indep. vectors and $\mathcal{B}_1$ is a spanning set

$$\Rightarrow |\mathcal{B}_2| \leq |\mathcal{B}_1| \quad ② \quad \xrightarrow{①+②} |\mathcal{B}_1| = |\mathcal{B}_2| \quad \boxtimes$$

Thm: If U is a subspace of V then $\dim(U) \leq \dim(V)$. If $\dim(U) = \dim(V)$, then $U = V$.

Two useful ideas: (If $\dim V = n$)
① any collection of n linearly independent vectors is a basis. (i.e. span V).
② any spanning set of size n is a basis (i.e. must be linearly independent)

Ex: What are possible subspaces of $\mathbb{R}^3$?
$\dim(\mathbb{R}^3) = 3$ so by Thm,

dim	Basis	vectors in subspace	geometric description
0	$\emptyset$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	origin
1	$\begin{pmatrix}a\\b\\c\end{pmatrix}$	$k\begin{pmatrix}a\\b\\c\end{pmatrix}$	line in direction $\langle a,b,c \rangle$ through origin
2	$\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}d\\e\\f\end{pmatrix}\right\}$ not $= k\begin{pmatrix}a\\b\\c\end{pmatrix}$	$k_1\begin{pmatrix}a\\b\\c\end{pmatrix} + k_2\begin{pmatrix}d\\e\\f\end{pmatrix}$	plane through origin containing $\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}d\\e\\f\end{pmatrix}\right\}$
3	$\{e_1, e_2, e_3\}$	$\left\{\begin{pmatrix}a\\b\\c\end{pmatrix} \middle\vert\, a, b, c \in \mathbb{R}\right\}$	$\mathbb{R}^3$