Lemma Let V be a vector space and $S \subset V$ where $S = \{v_1, \dots, v_n\}$. Then S is linearly dependent iff there is some vector in S, v_j so that $v_j = a_1 v_1 + \dots + 0 v_j + \dots + a_n v_n$

$\Rightarrow v_j \in \text{span}(\{v_1, \dots, \hat{v_j}, \dots, v_n\})$

Ex: Let $v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $v_4 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

is $\{v_1, v_2, v_3, v_4\}$ linearly independent?

Solution: Let $a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$ & solve for the a_i (if all $=0$, L.I.)

$$a_1\begin{pmatrix}1\\0\\1\end{pmatrix} + a_2\begin{pmatrix}0\\1\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\0\end{pmatrix} + a_4\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \longrightarrow \begin{cases} a_1 + a_3 + a_4 = 0 & \rightarrow a_4 = 2a_2 \\ a_2 + a_3 = 0 \\ a_1 + a_2 = 0 \end{cases} \quad a_2 = -a_3 = -a_1$$

Then we have $a_1 = a_3 = -a_2$ and $a_4 = 2a_2$

Let $a_2 = t$ (parameter) Set $t=1$: $-\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ Nontrivial solution ∴ linearly dependent

Lemma $\rightarrow$ $v_1 = v_2 - v_3 + 2v_4$ or $v_2 = v_1 + v_3 - 2v_4$.

What if we trim down our set (Thm)? [Q] Are v_1, v_2, v_3 linearly independent?

Sol

$$a_1\begin{pmatrix}1\\0\\1\end{pmatrix} + a_2\begin{pmatrix}0\\1\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \longrightarrow \begin{cases} a_1 + a_3 = 0 \\ a_2 + a_3 = 0 \\ a_1 + a_2 = 0 \end{cases}$$

$\rightarrow a_1 + a_3 = a_2 + a_3 \Rightarrow a_1 = a_2$

$\rightarrow 2a_1 = 0 \Rightarrow a_1 = 0$.

or:

$$\begin{matrix} a_1 + a_3 = 0 - \varepsilon_1 \\ a_2 + a_3 = 0 - \varepsilon_2 \\ a_1 + a_2 = 0 - \varepsilon_3 \end{matrix} \xrightarrow{-\varepsilon_1 + \varepsilon_3} \begin{cases} a_1 + a_3 = 0 \\ a_2 + a_3 = 0 \\ a_2 - a_3 = 0 \end{cases} \xrightarrow{-\varepsilon_2 + \varepsilon_3} \begin{cases} a_1 + a_3 = 0 \\ a_2 + a_3 = 0 \\ -2a_3 = 0 \end{cases} \rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{matrix}$$

Then v_1, v_2, v_3 are L.I. (so $\emptyset$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$ all L.I by Thm)

[Q] What is span(v_1, v_2, v_3, v_4)? By thm, same as span(v_1, v_2, v_3).

Same question as which vectors $\begin{pmatrix}x\\y\\z\end{pmatrix}$ are linear combinations of v_1, v_2, v_3?

$$\begin{matrix} a_1 + a_3 = x \\ a_2 + a_3 = y \\ a_1 + a_2 = z \end{matrix} \xrightarrow{-\varepsilon_1 + \varepsilon_3} \begin{matrix} a_1 + a_3 = x \\ a_2 + a_3 = y \\ a_2 - a_3 = z - x \end{matrix} \xrightarrow{-\varepsilon_2 + \varepsilon_3} \begin{matrix} a_1 + a_3 = x \\ a_2 + a_3 = y \\ -2a_3 = z - x - y \end{matrix} \quad a_3 = \frac{z-x-y}{2}$$

Notice, for any x, y, z there will always be a solution for a_1, a_2, a_3

$\Rightarrow$ span$(v_1, v_2, v_3) = \mathbb{R}^3$

Ex Determine a spanning set for $T = \left\{ \begin{pmatrix}-5s+t\\5s-6t\\s\\3t\end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$

Solution: We can think about "breaking apart" this general vector to make it a linear combination of specific vectors $\leftarrow$ they will be our spanning set

$$\begin{pmatrix}-5s+t\\5s-6t\\s\\3t\end{pmatrix} = s\begin{pmatrix}-5\\5\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\-6\\0\\3\end{pmatrix}$$

∴ one spanning set for T is $\left\{ \begin{pmatrix}-5\\5\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-6\\0\\3\end{pmatrix} \right\}$

To think about (on your own): Is T a vector space? If yes, prove it in the most effective (quickest) way possible. If no, state why.