Solutions to Written HW #1

① Given $\alpha = a+bi \in \mathbb{C} \neq 0$ find $c+di$ s.t. $c+di = \frac{1}{a+bi}$

<u>Solution:</u> $(c+di)(a+bi) = 1 \Rightarrow ac - bd + adi + bci = 1 = 1 + 0i$

$\Rightarrow ac - bd = 1$

$ad + bc = 0$

Ⓐ Assume $a \neq 0 \Rightarrow d = \frac{-bc}{a} \Rightarrow ac - b\left(\frac{-bc}{a}\right) = 1$

$\Rightarrow c\left(a + \frac{b^2}{a}\right) = 1$

$\Rightarrow c\left(\frac{a^2+b^2}{a}\right) = 1$

$\Rightarrow \boxed{c = \frac{a}{a^2+b^2}}$

$d = \frac{-b\left(\frac{a}{a^2+b^2}\right)}{a}$

$\boxed{d = \frac{-b}{a^2+b^2}}$

Ⓑ If $a = 0$, $-bd = 1 \longrightarrow \boxed{d = \frac{1}{-b}}$

$bc = 0 \longrightarrow b \neq 0 \Rightarrow \boxed{c = 0}$

Ⓒ If $b = 0$, then $\alpha \in \mathbb{R}$ so $\frac{1}{a} = c + di \Rightarrow \boxed{\frac{1}{a} = c,\ d = 0}$

② a) NOT a vector space. Example of axiom: not closed under addition

$\begin{pmatrix} t+1 \\ 2t \end{pmatrix} + \begin{pmatrix} t+1 \\ 2t \end{pmatrix} = \begin{pmatrix} 2t+2 \\ 4t \end{pmatrix} \neq 2t+1$
$= 2(2t)$

$\boxed{\text{OR}}$ many possibilities

additive identity:

$\begin{pmatrix} t+1 \\ 2t \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} t+1 \\ 2t \end{pmatrix}$

$t+1+z_1 = t+1 \Rightarrow z_1 = 0$
$2t + z_2 = 2t \Rightarrow z_2 = 0$ BUT $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin V$

b) NOT a vector space. Example: $\oplus$ not commutative.

$f \oplus g = f \circ g \neq g \circ f = g \oplus f$ in general.

c) Vector space.

- $+$ is closed: Let $p = a_1 + b_1x + c_1x^2$ $q = a_2 + b_2x + c_2x^2$.

  $p+q = (a_1+a_2) + (b_1+b_2)x + (c_1+c_2)x^2 \in P_2$ ✓

- $(\cdot)$ is closed: Let $p = a_1 + b_1x + c_1x^2$, $\lambda \in \mathbb{R}$

  $\lambda p = \lambda(a_1 + b_1x + c_1x^2) = (\lambda a_1) + (\lambda b_1)x + (\lambda c_1)x^2 \in P_2$ ✓

- $+$ commutative: $p+q$ above...

  $(a_1+a_2) + (b_1+b_2)x + (c_1+c_2)x^2 = (a_2+a_1) + (b_2+b_1)x + (c_2+c_1)x^2$
  $= (a_2 + b_2x + c_2x^2) + (a_1 + b_1x + c_1x^2)$
  $= q + p$ ✓

- $+$ is associative: $p+q$ from above plus $f = a_3 + b_3x + c_3x^2$

  $(p+q) + f = \left((a_1+a_2) + (b_1+b_2)x + (c_1+c_2)x^2\right) + a_3 + b_3x + c_3x^2$
  $= \left((a_1+a_2)+a_3\right) + \left((b_1+b_2)+b_3\right)x + \left((c_1+c_2)+c_3\right)x^2$
  since $+ \in \mathbb{R}$ is assoc. $= \left(a_1 + (a_2+a_3)\right) + \left(b_1 + (b_2+b_3)\right)x + \left(c_1 + (c_2+c_3)\right)x^2$
  $= a_1 + b_1x + c_1x^2 + (a_2+a_3) + (b_2+b_3)x + (c_2+c_3)x^2 = p + (q+f)$ ✓

• $(\cdot)$ is associative: Let $\lambda, \alpha \in \mathbb{R}$, p as above.

$\lambda(\alpha \cdot p) = \lambda \cdot \alpha(a_1 + b_1 x + c_1 x^2)$

$= \lambda \cdot ((\alpha a_1) + (\alpha b_1)x + (\alpha c_1)x^2)$

• commutative in $\mathbb{R}$

$= \lambda(\alpha a_1) + \lambda(\alpha b_1)x + \lambda(\alpha c_1)x^2 = (\lambda\alpha)a_1 + (\lambda\alpha)b_1 x + (\lambda\alpha)c_1 x^2 = (\lambda\alpha)\cdot p$ ✓

• additive identity: $\exists\, 0 \in P_2$ s.t. $0 + p = p + 0 = p \;\forall p \in P_2$.

Let $0 = 0 + 0x + 0x^2$. Then $0 + p = (0 + 0x + 0x^2) + (a_1 + b_1 x + c_1 x^2)$

$= (0 + a_1) + (0 + b_1)x + (0 + c_1)x^2$

$= a_1 + b_1 x + c_1 x^2 = p$ ✓

• additive inverse: Let $p \in P_2$. Want to find $-p$: (say q)

$p + q = q + p = 0$

$(a_1 + b_1 x + c_1 x^2) + (a_2 + b_2 x + c_2 x^2) = 0 + 0x + 0x^2$

$(a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 = 0 + 0x + 0x^2$

$\left.\begin{array}{l} \text{ww}: a_1 + a_2 = 0 \Rightarrow a_2 = -a_1 \\ \text{—}: b_1 + b_2 = 0 \Rightarrow b_2 = -b_1 \\ \sim: c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \end{array}\right\} -p = -a_1 - b_1 x - c_1 x^2$

• Multiplicative Identity: $1 \cdot p = p \;\forall p \in P_2$?

$1 \cdot p = 1 \cdot (a_1 + b_1 x + c_1 x^2) = 1 \cdot a_1 + 1 \cdot b_1 x + 1 \cdot c_1 x^2 = a_1 + b_1 x + c_1 x^2 = p$ ✓

• Distributive properties:

① $\alpha(p + q) = \alpha p + \alpha q$? $\forall \alpha \in \mathbb{R}$, $p, q \in P_2$

$\alpha((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) = \alpha(a_1 + a_2) + \alpha(b_1 + b_2)x + \alpha(c_1 + c_2)x^2$

Real #s distribute $= \alpha a_1 + \alpha a_2 + (\alpha b_1 + \alpha b_2)x + (\alpha c_1 + \alpha c_2)x^2$

$= (\alpha a_1 + \alpha b_1 x + \alpha c_1 x^2) + (\alpha a_2 + \alpha b_2 x + \alpha c_2 x^2)$

$= \alpha(a_1 + b_1 x + c_1 x^2) + \alpha(a_2 + b_2 x + c_2 x^2)$

$= \alpha p + \alpha q$ ✓

② $(\alpha + \lambda)p = \alpha p + \lambda p$? $\forall \alpha, \lambda \in \mathbb{R}$, $p \in P_2$

$(\alpha + \lambda)\cdot p = (\alpha + \lambda)\cdot(a_1 + b_1 x + c_1 x^2)$

$= (\alpha + \lambda)a_1 + (\alpha + \lambda)b_1 x + (\alpha + \lambda)c_1 x^2$

Real #s distribute $= \alpha a_1 + \lambda a_1 + (\alpha b_1 + \lambda b_1)x + (\alpha c_1 + \lambda c_1)x^2$

$= (\alpha a_1 + \alpha b_1 x + \alpha c_1 x^2) + (\lambda a_1 + \lambda b_1 x + \lambda c_1 x^2)$

$= \alpha(a_1 + b_1 x + c_1 x^2) + \lambda(a_1 + b_1 x + c_1 x^2)$

$= \alpha p + \lambda p$ ✓

Therefore, P_2 is a vector space over $\mathbb{R}$. ☒

③ Let $x, y \in V$, $a, b \in \mathbb{R}$.

$(a+b)(x+y) \underset{\uparrow}{=} (a+b)x + (a+b)y \underset{\uparrow}{=} ax + bx + ay + by$ ☒

• distributes over +

scalar addition distributes over +

4) a) • Is $0 \in W$? $\qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, 4x_1 + 3x_2 - 2x_3 = 0 \text{ and } x_1 - x_3 = 0 \right\}$.

$0 \in \mathbb{R}^3$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. $\quad 4(0) + 3(0) - 2(0) = 0 \checkmark \quad 0 - 0 = 0 \checkmark \quad$ yes ✓

• Is W closed under +?

Let $x, y \in W$. Then $4x_1 + 3x_2 - 2x_3 = 0$, $x_1 - x_3 = 0$, $4y_1 + 3y_2 - 2y_3 = 0$, and $y_1 - y_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \underset{\text{in } \mathbb{R}^3}{=} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

Check:

$$4(x_1 + y_1) + 3(x_2 + y_2) - 2(x_3 + y_3)$$
$$= 4x_1 + 4y_1 + 3x_2 + 3y_2 - 2x_3 - 2y_3$$
$$= \underbrace{(4x_1 + 3x_2 - 2x_3)}_{\substack{=0 \\ \text{since } x \in W}} + \underbrace{(4y_1 + 3y_2 - 2y_3)}_{\substack{=0 \\ \text{since } y \in W}} = 0 \checkmark$$

$$(x_1 + y_1) - (x_3 + y_3) = x_1 + y_1 - x_3 - y_3$$
$$= \underbrace{(x_1 - x_3)}_{\substack{=0 \\ \text{since } x \in W}} + \underbrace{(y_1 - y_3)}_{\substack{=0 \\ \text{since } y \in W}} = 0 \checkmark$$

Therefore, $x + y \in W$ so W closed under +.

• Is W closed under •? Let $a \in \mathbb{R}$, $x \in W$.

Then $a \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \underset{\text{in } \mathbb{R}^3}{=} \begin{pmatrix} ax_1 \\ ax_2 \\ ax_3 \end{pmatrix}$ and since $x \in W$, $4x_1 + 3x_2 - 2x_3 = 0$ and $x_1 - x_3 = 0$.

$$\Rightarrow a(4x_1 + 3x_2 - 2x_3) = a \cdot 0 = 0 \qquad a(x_1 - x_3) = a \cdot 0 = 0$$
$$\Rightarrow 4ax_1 + 3ax_2 - 2ax_3 = 0 \checkmark \qquad ax_1 - ax_3 = 0 \checkmark$$

Since $0 \in W$, W is closed under + and •, W is a subspace of $\mathbb{R}^3$. ☒

b) Not a subspace → not closed under •

Ex $\frac{1}{2} \in \mathbb{R}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in W$. But $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \notin W$ since $\frac{1}{2} \notin \mathbb{Z}$.

5) Since $T \subset P_2$, we need only check if $0 \in T$, T is closed under + and T is closed under •. The 0 from P_2 is $0 + 0x + 0x^2$.

$$\int_0^2 0 + 0t + 0t^2 \, dt = \int_0^2 0 \, dt = 0\Big]_0^2 = 0 \checkmark$$

Let $p, q \in T$. Then $\int_0^2 p(t)\,dt = 0$ and $\int_0^2 q(t)\,dt = 0$.

Consider $p+q$. $\displaystyle\int_0^2 (p+q)(t)\,dt \underset{\text{def of } +}{=} \int_0^2 p(t) + q(t)\,dt \underset{\text{properties of } \int}{=} \int_0^2 p(t)\,dt + \int_0^2 q(t)\,dt = 0 + 0 = 0$

So $p + q \in T$.

Consider ap where $a \in \mathbb{R}$. $\displaystyle\int_0^2 (ap)(t)\,dt \underset{\text{def of } \cdot}{=} \int_0^2 a(p(t))\,dt \underset{\text{properties of } \int}{=} a\int_0^2 p(t)\,dt = a \cdot 0 = 0$

So $ap \in T$.

Therefore, T is a vector space since it is a subspace of P_2. ☒

(6) • $0 \in W$? $W = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| a_{11} = a_{22} = 0,\ a_{12} = -a_{21} \right\}$

$0 \in M_2(\mathbb{R}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $0 = 0 = 0$ ✓ $0 = -0$ ✓

• W closed under $+$? Let $A, B \in W$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \underset{\text{def of + in } M_2(\mathbb{R})}{=} \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

Since $a_{11} = a_{22} = 0$
and $+\ b_{11} = b_{22} = 0$

$a_{11}+b_{11} = a_{22}+b_{22} = 0$ ✓

Since $a_{12} = -a_{21}$
and $+\ b_{12} = -b_{21}$

$a_{12}+b_{12} = -(a_{21}+b_{21})$ ✓

• W closed under $\cdot$? Let $\lambda \in \mathbb{R}$, $A \in W$.

$$\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underset{\text{def of } \cdot \text{ in } M_2(\mathbb{R})}{=} \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$

We know $a_{11} = a_{22} = 0$

$\Rightarrow \lambda a_{11} = \lambda a_{22} = \lambda \cdot 0 = 0$ ✓

We know $a_{12} = -a_{21}$

Multiply both sides by λ: $\lambda a_{12} = -\lambda a_{21}$ ✓

Therefore, W is a subspace of $M_2(\mathbb{R})$ ☒