Notation: "if and only if" = "iff" or $\Longleftrightarrow$

Proof: If U is a subspace of V ($\Rightarrow$) then the three conditions are satisfied

($\Leftarrow$) Suppose $U \subset V$ and satisfies the three conditions. We need to prove U is a vector space.

By assumption, $0 \in U$, U is closed under $+$, and U is closed under $\cdot$

commutativity of $+$, associativity of $+$, associativity of $\cdot$, multiplicative identity all hold in U since they hold in the larger space V.

Additive Inverse? If $u \in U$, we need $-u \in U$.

By assumption, U is closed under scalar multiplication. So $(-1)\cdot u \in U$. Then by Prop 5, $-u \in U$.

$\Rightarrow U$ is a vector space inside of V so it is a subspace of V. $\boxtimes$

Ex: Let $V = \{f: \mathbb{R} \to \mathbb{R} \mid f(x) = f(-x)\ \forall x \in \mathbb{R}\}$. Is this a vector space? (vectors = functions) vector space ($V \subset \mathbb{R}^{\mathbb{R}}$)

GOAL: Find a bigger vector space that V lies in & use the claim

① Is $0 \in V$? $0 \in \mathbb{R}^{\mathbb{R}}$ is the zero function (spits out 0, no matter what input)

$\Rightarrow f(x) = f(-x)\ \forall x \in \mathbb{R}$ where $f = 0$.

② Is V closed under $+$? Let $f, g \in V$, is $f+g \in V$?

We know $f(x) = f(-x)$ and $g(x) = g(-x)$ since $f, g \in V$ $\forall x \in \mathbb{R}$.

Check: $(f+g)(x) \underset{\substack{\uparrow \\ \text{addition of fns}}}{=} f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x)$ yes!

③ Is V closed under $\cdot$? Let $f \in V$, $a \in \mathbb{R}$. Is $a \cdot f \in V$?

Check: $(a\cdot f)(x) = a\cdot(f(x)) = a\cdot(f(-x)) = (a\cdot f)(-x)$ yes! So V is a vector space since it's a subspace of $\mathbb{R}^{\mathbb{R}}$

Ex: $V = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} a,b,c,d \in \mathbb{R} \\ \text{and } a+d=0 \end{matrix}\right\}$ (vectors = matrices) Is this a vector space?

Strategy: Find bigger v. space containing & use proposition.

$V \subseteq M_2(\mathbb{R})$

① Is $0 \in V$? $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $0+0 \overset{?}{=} 0$ ✓

② If $A, B \in V$, is $A+B \in V$? $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ $a_1, b_1, \dots, c_2, d_2 \in \mathbb{R}$, $a_1 + d_1 = 0 = a_2 + d_2$

$A+B = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$ all entries still real.

$(a_1+a_2) + (d_1+d_2) = (a_1+d_1) + (a_2+d_2) = 0+0 = 0$ ✓ so $A+B \in V$

③ If $A \in V$, $k \in \mathbb{R}$, is $k\cdot A \in V$? (A as above)

$kA = k\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{pmatrix}$ all real ✓

$ka_1 + kd_1 = k(a_1+d_1) = k\cdot 0 = 0$ ✓ $kA \in V$

So V is a vector space.