One more funky one: (for you!) $V=\mathbb{R}^2$, define $u\oplus v = \begin{pmatrix}u_1\\u_2\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}u_1+v_1+1\\u_2+v_2+1\end{pmatrix}$, $a\odot v = a\odot\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}av_1+a-\\av_2+a-\end{pmatrix}$

Check, is this a vector space over $\mathbb{R}$?

Both are closed ✓

Commutativity of $\oplus$: $\begin{pmatrix}u_1\\u_2\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}u_1+v_1+1\\u_2+v_2+1\end{pmatrix} = \begin{pmatrix}v_1+u_1+1\\v_2+u_2+1\end{pmatrix} = \begin{pmatrix}v_1\\v_2\end{pmatrix}\oplus\begin{pmatrix}u_1\\u_2\end{pmatrix}$ ✓

associativity of $\oplus$: $\left(\begin{pmatrix}u_1\\u_2\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)\oplus\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}u_1+v_1+1\\u_2+v_2+1\end{pmatrix}\oplus\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}u_1+v_1+w_1+2\\u_2+v_2+w_2+2\end{pmatrix} \overset{\text{check}}{=} u\oplus(v\oplus w)$

associativity of $\odot$: $a\odot\left(b\odot\begin{pmatrix}v_1\\v_2\end{pmatrix}\right) = a\odot\begin{pmatrix}bv_1+b-1\\bv_2+b-1\end{pmatrix} = \begin{pmatrix}a(bv_1+b-1)+a-1\\a(bv_2+b-1)+a-1\end{pmatrix} = \begin{pmatrix}abv_1+ab-1\\abv_2+ab-1\end{pmatrix} = (ab)\odot v$ ✓

additive identity? 0 vector?

Say $\begin{pmatrix}z_1\\z_2\end{pmatrix} \overset{\text{WANT}}{\Longrightarrow} \underbrace{\begin{pmatrix}u_1\\u_2\end{pmatrix}\oplus\begin{pmatrix}z_1\\z_2\end{pmatrix}}_{} = \begin{pmatrix}u_1\\u_2\end{pmatrix}$

not usual!

$\begin{pmatrix}u_1+z_1+1\\u_2+z_2+1\end{pmatrix} = \begin{pmatrix}u_1\\u_2\end{pmatrix} \Longrightarrow \begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$

additive inverse? Want $\begin{pmatrix}v_1\\v_2\end{pmatrix}$ so that $u\oplus v = 0$ careful... $\begin{pmatrix}u_1\\u_2\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$

Let's play... $\begin{pmatrix}u_1\\u_2\end{pmatrix}\oplus\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}u_1+v_1+1\\u_2+v_2+1\end{pmatrix} \overset{\text{WANT}}{=} \begin{pmatrix}-1\\-1\end{pmatrix} \Longrightarrow \begin{matrix}u_1+v_1+1=-1\\u_2+v_2+1=-1\end{matrix} \Longrightarrow \begin{matrix}v_1=-2+-u_1\\v_2=-2+-u_2\end{matrix}$

Then we write $-\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}-u_1-2\\-u_2-2\end{pmatrix}$

Multiplicative identity: Is $1\odot v = v$? $1\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}v_1+1-1\\v_2+1-1\end{pmatrix} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ ✓

On your own, check distributive properties: $a\odot(u\oplus v) \overset{?}{=} a\odot u\oplus a\odot v$

$(a+b)\odot v \overset{?}{=} a\odot v \oplus b\odot v$

I'll only make you do 1 or 2 for HW. Yes! Vector space.

Properties of Vector Spaces

Prop 1: The additive identity (0 vector) is unique.

→ Proof in Axler

Prop 2: Every element has a unique additive inverse.

Proof: By definition of vector space, every vector $v\in V$ has an additive inverse, say u. Suppose it has another, u' (u is not unique)

GOAL: Show $u=u'$. (Use what we know to get there)

We know $u+v=0$ and $u'+v=0$. We don't "know" subtraction.

So, $u = u+0 = u+(v+u') = (u+v)+u' = 0+u' = u'$

additive identity of V; u' is an additive inverse; associativity of +; u's add inverse; additive identity of V

Therefore $u=u'$, so u is unique ☒

Notation: Since additive inverses are unique, we will denote it as $-v$. Then we have notation for subtraction: $u-v = u+(-v)$.

Prop 3: $0\cdot v = 0 \quad \forall v\in V$

Proof: Let $v\in V$. Then $0\cdot v = (0+0)\cdot v = 0\cdot v + 0\cdot v$ (0 = 0 in scalars; distribution) Now add $-0\cdot v$ to both sides: $\underbrace{0\cdot v+(-0\cdot v)}_{0} = 0\cdot v + \underbrace{0\cdot v+(-0\cdot v)}_{0}$, so $0 = 0\cdot v + 0$ ☒