* One vector never needs to be used $(2x^2-x)$

or

* One vector can be written as a linear combination of others

or $\quad 2x^2-x = 2(x^2) - 1(x) + 0(1)$

* Can write vectors as multiple linear combinations of the set

or $\quad ax^2+bx+c = \underline{0}\,x^2 + \underline{\tfrac{a}{2}+b}\,x + \underline{c}\,1 + \underline{\tfrac{a}{2}}(2x^2-x)$

* The zero vector can be written with some nonzero coefficients

$$0 = \underline{-2}\,x^2 + \underline{1}\,x + \underline{0}\,1 + \underline{1}\,(2x^2-x)$$

The * are all equivalent and give us a new concept:

**Def:** Let $V$ be a vector space over $F$ and let $S=\{v_1,\dots,v_m\}$ be a set of vectors from $V$. $S$ (or the vectors in $S$) are called *linearly independent* if the only choice of $a_1,\dots,a_m \in F$ that makes $a_1v_1+\dots+a_mv_m=0$ is $a_1=\dots=a_m=0$.

↑ we call this the trivial solution.

If the list of vectors $S$ is not linearly independent, we say they are *linearly dependent*

To show, find $a_1,\dots,a_m$ not all zero so $a_1v_1+\dots+a_mv_m=0$.

**Ex:** Are the following linearly independent?

$\begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\1\\3\end{pmatrix}$ i.e. Can we solve $\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = a\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + b\begin{pmatrix}0\\1\\1\\2\end{pmatrix} + c\begin{pmatrix}1\\1\\1\\3\end{pmatrix}$ nontrivially? If no, lin. indep

Play! $\Rightarrow$
$0 = a+c \Rightarrow a=-c$
$0 = b+c \Rightarrow b=-c$
$\Big\}\ -2c+c=0 \Rightarrow c=0 \Rightarrow a=0,\ b=0.$
$0 = a+b+c$
$0 = 2a+2b+3c \rightarrow -2c+-2c+3c=0 \Rightarrow c=0$

yes!

**Ex:** Are these linearly independent?

$\begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\\2\end{pmatrix}, \begin{pmatrix}-2\\1\\-1\\-2\end{pmatrix}$ i.e. Is the only solution to $\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = a\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + b\begin{pmatrix}0\\1\\1\\2\end{pmatrix} + c\begin{pmatrix}-2\\1\\-1\\-2\end{pmatrix}$ the trivial one?

$0 = a-2c \rightarrow a=2c$
$0 = b+c \rightarrow b=-c$
$0 = a+b-c \rightarrow 2c-c-c=0$
$0 = 2a+2b-2c \rightarrow 4c-2c-2c=0.$

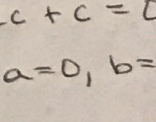
No: $\begin{pmatrix}-2\\1\\-1\\-2\end{pmatrix} = \begin{pmatrix}0\\1\\1\\2\end{pmatrix} - 2\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$

let $c=1$

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = 2\begin{pmatrix}1\\0\\1\\2\end{pmatrix} - \begin{pmatrix}0\\1\\1\\2\end{pmatrix} + \begin{pmatrix}-2\\1\\-1\\-2\end{pmatrix}$$

**Ex:** Can you find a linearly independent set in $M_2(\mathbb{R})$?

$\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ or any subset!



**Thm:** Let $V$ be a vector space and $T \subseteq S \subseteq V$. Then

① if $T$ is a set of linearly dependent vectors from $V$, then $S$ is also linearly dependent

② if $S$ is linearly independent, then $T$ is also linearly independent

③ if span$(T)=V$, then span$(S)=V$ (and if $S \neq T$, then $S$ is linearly dependent.