Notice, we can rewrite the condition in $V$ as $d=-a$. Then all vectors are of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ or $a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

scalars vectors

**Def:** Let $S=\{v_1, v_2, \dots, v_n\}$ be a set of vectors from a given vector space $V$ over $F$. A <u>linear combination</u> of vectors from $S$ is any vector $v$ that can be written as $v = a_1v_1 + a_2v_2 + \dots + a_nv_n$ where $a_1, a_2, \dots, a_n \in F$. Notation: $\sum_{i=1}^{n} a_i v_i$

**Ex:** In $\mathbb{R}^3$, $S=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. Is $\begin{pmatrix}14\\-2\\3\end{pmatrix}$ a linear combination of vectors in $S$?
$\qquad e_1, e_2, e_3$

Easy to see → $14\begin{pmatrix}1\\0\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix} = 14e_1 + (-2)e_2 + 3e_3$. yes.

**Ex:** In $P_2$, $S=\{2, (1-x), x^2\}$. Is $2x^2-3x+4$ a linear combination of vectors in $S$?

$2x^2-3x+4 = \underline{2}(x^2) + \underline{3}(1-x) + \underline{\tfrac{1}{2}}(2)$

$= ax^2 + b(1-x) + c(2) \Rightarrow \overset{2}{\overset{\shortparallel}{a}}x^2 - \overset{3}{\overset{\shortparallel}{b}}x + \overset{4}{\overset{\shortparallel}{\overbrace{(b+2c)}}}$

**Ex:** In $M_2(\mathbb{R})$, $S=\left\{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -5 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 5 & 0 \end{pmatrix}\right\}$ Is $\begin{pmatrix} 1 & 0 \\ 0 & \boxed{1} \end{pmatrix}$ a linear combination of vectors in $S$?

No!

There's no way to get a nonzero number in 2,2 entry.

**For you:** $S=\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$
$\qquad v_1, v_2, v_3$

1. Can you write $\begin{pmatrix}3\\6\\7\end{pmatrix}$ as a linear combination of $v_1, v_2, v_3$?

2. Can you write any vector in $\mathbb{R}^3$ as a linear combination of $v_1, v_2, v_3$?

**Solution** ① $\begin{pmatrix}3\\6\\7\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow$
$3 = a+b+c \Rightarrow c=-3$
$6 = a+b \Rightarrow a=-1$
$7 = b$

$= -1\begin{pmatrix}1\\1\\0\end{pmatrix} + 7\begin{pmatrix}1\\1\\1\end{pmatrix} + -3\begin{pmatrix}1\\0\\0\end{pmatrix}$

② Let $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$ $\begin{pmatrix}x\\y\\z\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow$
$x = a+b+c \Rightarrow c = x-y$
$y = a+b \Rightarrow a = y-z$
$z = b$

$= (y-z)\begin{pmatrix}1\\1\\0\end{pmatrix} + z\begin{pmatrix}1\\1\\1\end{pmatrix} + (x-y)\begin{pmatrix}1\\0\\0\end{pmatrix}$

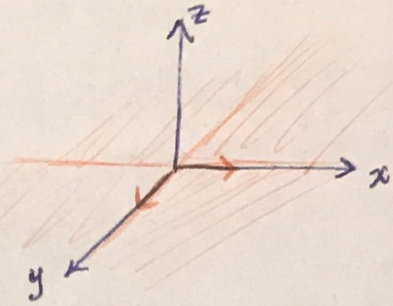
**Def:** Let $V$ be a vector space over $F$ and let $S=\{v_1, v_2, \dots, v_n\} \subseteq V$. The set of all linear combinations of $S$ is called the <u>span</u> of $S$ and is written
$\text{span}(S) = \text{span}(v_1, v_2, \dots, v_n) = \{v \mid v = a_1v_1 + a_2v_2 + \dots + a_nv_n \text{ where } a_i \in F \text{ for } i=1,2,\dots,n\}$
If $\text{span}(S)=V$, we say $S$ spans $V$ or $V$ is spanned by $S$ or $S$ is a spanning set for $V$.

Visualizing some examples

- What is $\operatorname{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$?

$$=\left\{\underbrace{a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\begin{pmatrix}a\\b\\0\end{pmatrix}} \;\middle|\; a,b\in\mathbb{R}\text{ (or }\mathbb{C})\right\}$$

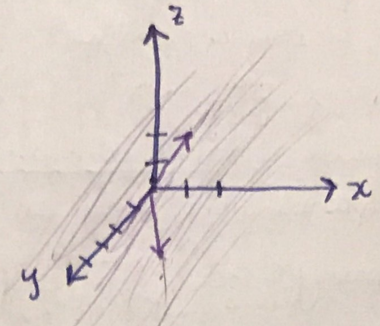
$\begin{pmatrix}a\\b\\0\end{pmatrix} \leftarrow$ xy plane.




- What is $\operatorname{span}\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\4\\1\end{pmatrix}\right)$?

$$=\left\{a\begin{pmatrix}1\\0\\2\end{pmatrix}+b\begin{pmatrix}2\\4\\1\end{pmatrix} \;\middle|\; a,b\in\mathbb{R}\text{ (or }\mathbb{C})\right\}.$$

all vectors of the form $\begin{pmatrix}a+2b\\4b\\2a+b\end{pmatrix}$




- What is $\operatorname{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right)$?

Line through origin $x=\frac{y}{2}=\frac{z}{3}$

or all vectors $\begin{pmatrix}a\\2a\\3a\end{pmatrix}$ or $\begin{matrix}x=t\\y=2t\\z=3t\end{matrix}$

- $\operatorname{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right)$? $=\begin{pmatrix}a\\0\\0\end{pmatrix}+\begin{pmatrix}2b\\0\\0\end{pmatrix}=\begin{pmatrix}a+2b\\0\\0\end{pmatrix}=\begin{pmatrix}c\\0\\0\end{pmatrix}$ x axis.

- $\operatorname{span}(1,x,x^2)$? $P_2$.

**Theorem**: Let $V$ be a vector space and $S\subseteq V$ where $S=\{v_1,\dots,v_n\}$. Then

1) $\operatorname{span}(S)$ is a subspace of $V$
2) $\operatorname{span}(S)$ is the smallest subspace of $V$ containing $S$.

**Proof**: 1) We use the subspace proposition, First, we must prove $0\in\operatorname{span}(S)$.

$0=a_1v_1+\dots+a_nv_n$ if $a_1=\dots=a_n=0$. ✓

Let $u,v\in\operatorname{span}(S)$ be arbitrary. We wish to show $u+v\in\operatorname{span}(S)$.

Since $u,v\in\operatorname{span}(S)$, $u=a_1v_1+\dots+a_nv_n$, $v=b_1v_1+\dots+b_nv_n$

$u+v=(a_1v_1+\dots+a_nv_n)+(b_1v_1+\dots+b_nv_n)$

$=(a_1+b_1)v_1+\dots+(a_n+b_n)v_n \in\operatorname{span}(S)$ ✓ because $V$ is a v. space

Finally, let $\lambda\in F$, $u\in\operatorname{span}(S)$ Then $\lambda\cdot u=\lambda\cdot(a_1v_1+\dots+a_nv_n)$

$=(\lambda a_1)v_1+\dots+(\lambda a_n)v_n \in\operatorname{span}(S)$ ✓ because $V$ is a v. space

Then $\operatorname{span}(S)$ is a subspace of $V$.

2) We know $\operatorname{span}(S)$ contains each $v_i$ in $S$ (set $a_i=1$, rest 0).

To show it is the smallest, suppose we have another subspace of $V$ containing $S$, called $U$. We want to show $\operatorname{span}(S)\subseteq U$.

Let $u\in\operatorname{span}(S)$ be arbitrary. Then $u=a_1v_1+\dots+a_nv_n$

Since $U$ is a vector space containing $v_1,\dots,v_n$ and vector spaces are closed under addition & scalar multiplication, $u\in U$.

Then $\operatorname{span}(S)\subseteq U$ and so $\operatorname{span}(S)$ is the smallest subspace containing $S$. ☒