Def: Addition on a set V is a function that takes two elements of a set, say u and v, and sends them to another element of the set, u+v. ($+: V\times V \to V$, $u,v \mapsto u+v$)

Def: Scalar multiplication on a set V over some other set (e.g. $\mathbb{R}$ or $\mathbb{C}$), is a function that takes a scalar from $\mathbb{R}$ or $\mathbb{C}$ and an element from V to produce an element in V. Ex: $\lambda \in \mathbb{R}$, $v \in V \Rightarrow \lambda \cdot v = \lambda v \in V$ ($\cdot: \mathbb{R}\times V \to V$, $\lambda, v \mapsto \lambda v$)

Def: A vector space V over a set F ($\mathbb{R}$ or $\mathbb{C}$ usu.) is a set with addition & scalar multiplication where the following hold:

("$\in$": is an element of; "$\forall$": "for every" or "for all")

- commutativity $u+v = v+u \ \forall u,v \in V$
- associativity $(u+v)+w = u+(v+w)$ & $a(bv) = (ab)v \ \forall u,v,w \in V \ \forall a,b \in F$
- additive identity $\exists\, 0 \in V$ s.t. $0+v = v+0 = v \ \forall v \in V$
- additive inverse $\forall v \in V \ \exists w \in V$ s.t. $v+w = w+v = 0$
- multiplicative identity $1 \cdot v = v \ \forall v \in V$
- distributive properties: $a(u+v) = au+av$ & $(a+b)v = av+bv \ \forall u,v \in V,\ a,b \in F$

Ex $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

Vectors? $(v_1, v_2, \ldots, v_n)$ where $v_i \in \mathbb{R}$. Scalars? All real numbers.

Think through the above PLUS → do addition and scalar multiplication make sense? If I add two elements of $\mathbb{R}^n$ will I get another? This is closure

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix}$$ since each $v_i + w_i \in \mathbb{R}$, this is in $\mathbb{R}^n$.

Similarly, $$r\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix} \in \mathbb{R}^n$$

Ex: additive inverse: Given $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ what is w so that $v+w = w+v = 0$? $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\Rightarrow w = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix}$$

Ex: $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ (OR over $\mathbb{R}$)

In $\mathbb{C}^2$: $$\begin{pmatrix} -1+4i \\ 2+3i \end{pmatrix} + \begin{pmatrix} 5-6i \\ -4+i \end{pmatrix} = \begin{pmatrix} 4-2i \\ -2+4i \end{pmatrix} \qquad 5\begin{pmatrix} -1+4i \\ 2+3i \end{pmatrix} = \begin{pmatrix} -5+20i \\ 10+15i \end{pmatrix} \qquad (1-i)\begin{pmatrix} -4+i \\ 1+i \end{pmatrix} = \begin{pmatrix} -3+5 \\ 2 \end{pmatrix}$$

★ When showing closure → use arbitrary elements. Check through above on your own. (Axler 1A)

Ex In general, F^n is a vector space over F (for us, $F = \mathbb{R}$ or $\mathbb{C}$)

Non-ex $\mathbb{R}^2$ is NOT a vector space over $\mathbb{C}$ $(1-i)\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4+4i \\ 1-i \end{pmatrix}$ is not in $\mathbb{R}^2$ $\therefore$ not closed under $\cdot$

Ex $M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}$ set of all 2x2 matrices w/ real entries

$M_2(\mathbb{R})$ is a vector space over $\mathbb{R}$

Check a few: commutative ✓ since individual entry + is (same w/ associative).

What's the zero vector? ____ $+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ... ____ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Funky Ex Let $V = \mathbb{R}$ but redefine + and $\cdot$:

$\oplus: a \oplus b = a + 2b \ \forall a,b \in V$ (Ex: $2 \oplus 3 = 2+6 = 8$)

$\odot: a \odot v = av$ (same as usual).

(a: scalar, v: vector)

Is V a vector space over $\mathbb{R}$? $(1 \oplus 2) \oplus 3 = 5 \oplus 3 = 11$ $\quad 1 \oplus (2 \oplus 3) = 1 \oplus 8 = 17$ NO! $\oplus$ not assoc.