Similarly,

Prop 4: $a \cdot 0 = 0 \quad \forall a \in F$ (Proof in text)

Prop 5: $(-1) \cdot v = -v \quad \forall v \in V$ ← additive inverse of v

↑ scalar multiplication by -1

Proof: If $(-1)\cdot v$ is $-v$, then $v + (-1)\cdot v$ must be 0.

$v + (-1)\cdot v = 1 \cdot v + (-1)\cdot v = (1 + (-1))\cdot v = 0 \cdot v = 0$ by Prop 3. ☒

↑ multiplicative identity ↑ distribution

More Examples of Vector Spaces that we'll be using

Ex: Define $P_n = \{a_0 + a_1 x + \ldots + a_n x^n \mid a_0, \ldots, a_n \in \mathbb{R}\}$

= set of all polynomials of degree $\leq n$

P_n is a vector space over $\mathbb{R}$.

If $f, g \in P_n$, $f+g$? $\quad f = a_0 + a_1 x + \ldots + a_n x^n \quad g = b_0 + b_1 x + \ldots + b_n x^n$

$f + g = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$

Note: $f = \underline{x^3} - 2x^2 + 8 \quad g = \underline{-x^3} + 8x - 4 \rightarrow f + g = \underline{-2x^2} + 8x + 4$ need $\leq n$ for closed

on your own → go through checklist.

Scalar multiplication: $\lambda f = \lambda(a_0 + a_1 x + \ldots + a_n x^n) = \lambda a_0 + \lambda a_1 x + \ldots + \lambda a_n x^n$

Notation: If S is a set F^S denotes the set of functions from S to F ($F = \mathbb{R}$ or $\mathbb{C}$)

Ex $\mathbb{R}^{\mathbb{R}} = \{$functions from $\mathbb{R}$ to $\mathbb{R}\} \Rightarrow x^2 \in \mathbb{R}^{\mathbb{R}}$, $\sin(x) \in \mathbb{R}^{\mathbb{R}}$

$\mathbb{R}^{[0,1]}$ also has x^2 & $\sin(x)$. $\frac{1}{x-3} \in \mathbb{R}^{[0,1]}$ BUT! $\frac{1}{x-3} \notin \mathbb{R}^{\mathbb{R}}$

All vector spaces! vectors = functions here.

Ex $V = \left\{ \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ is this a vector space?

No! $2 \cdot \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2c & 2 \end{pmatrix} \notin V$ not closed under scalar multiplication.

Subspaces

What if we take a subset of a known vector space? Is it necessarily a vector space? No!

Def: A subset U of a vector space V is called a subspace of V if it is also a vector space using the same addition and scalar multiplication as V.

Proposition: Let V be a vector space over F and let $U \subset V$ "is a subset of"

Then U is a subspace of V if and only if

- $0 \in U$ (additive identity)
- U is closed under $+$: $\forall u, w \in U \Rightarrow u + w \in U$.
- U is closed under $\cdot$: $\forall a \in F, u \in U \Rightarrow au \in U$.

★ NOTE: This means (if true) we don't need to prove all the vector space axioms if we know the set is a subset of a known vector space ($\mathbb{R}^n$, $\mathbb{C}^n$, $M_n(\mathbb{R})$, P_n, etc.)

Notation: "if and only if" = "iff" or $\Longleftrightarrow$

Proof: If U is a subspace of V $(\Rightarrow)$ then the three conditions are satisfied.

$(\Leftarrow)$ Suppose $U \subset V$ and satisfies the three conditions. We need to prove U is a vector space.

By assumption, $0 \in U$, U is closed under $+$, and U is closed under $\cdot$

commutativity of $+$, associativity of $+$, associativity of $\cdot$, multiplicative identity all hold in U since they hold in the larger space V.

Additive Inverse? If $u \in U$, we need $-u \in U$.

By assumption, U is closed under scalar multiplication. So $(-1) \cdot u \in U$. Then by Prop 5, $-u \in U$.

$\Rightarrow U$ is a vector space inside of V so it is a subspace of V. $\boxtimes$